# MATHEMATICAL MODELING OF PROCESSES AND SYSTEMS

UDC 533.69.04:519.642 (045)

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## NUMERICAL SOLUTION OF WING INTEGRO-DIFFERENTIAL EQUATION

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***Abstract—****A numerical solution method of Prandtl integro-differential equation for a finite span wing was presented. Comparative calculations with Glauert's analitycal method were performed. A good agreement of results on the non-uniform mesh was obtained.*

**Index Terms—**Aerodynamics; circulation; lifting-line theory; vortex; wing.

## I. INTRODUCTION

Calculation methods of aircraft aerodynamic characteristics must represent physical phenomena and on the other hand must be simple, universal and authentic. It is better to have effective methods based on simplified algorithms at the precedent stage of computer aided design (CAD) systems process.

## II. REVIEW OF PUBLICATIONS

Till now there was no accurate solution in a common form for the Prandtl singular integro-differential equation. There were approximate methods only. The first who engaged in the equation was Betz. He considered a wing with the rectangular plan-form. Munk proved the minimal induced drag was observed for the wing with an elliptical circular distribution. Betz and Fuchs [1] wrote the desired solution in a form of the elliptical distribution multiplied to infinite polynomial series. The complicated calculations and slow convergence restricted the using of these approaches.

The Trefftz's approach turned out to be more effective and, developed lately by Glauert [2], began to be used in practice for wings with simple plan-forms. The Glauert's method still remains as basic one within the lifting-line theory. Its essence was in the expansion of circulation distribution in a form of infinite Fourier's series that converged quickly. In practice one kept the first several terms of series which may be obtained from solution of the linear algebraic equations system.

The Glauert's method was developed by such authors as Golubev, Lotz, Carafoli, Karamchetti, Multhopp, J. Anderson, R. Anderson, Burago, Nuzhin, Risberg, Yuriev and many others.

Efforts were made for developing the method and getting a common approach by means of taking into account of the plan-form. Carafoli [1], Karamchetti, Lotz used Fourier's series expansions for chord length and incidence over the span. Carafoli described a chord variation with three terms of series.

A lot of methods (Multhopp, Couethe and Chow, Bertin and Smith), as well as Glauert's one based on the collocation method, were made. Multhopp [3] presented an approach using the Gauss's quadrature formulae. Accuracy of collocation methods depends on the choice manner of points especially when a chord and an incidence have harshnesses along the span.

R. Anderson and Milsappe [4], Bera, Berbente passed over the difficulties of the collocation points choice using variational approach for determining the Fourier's expansion coefficients.

Monegato and Pennachietti [5] made a quadrature of the Prandtl equation with Chebyshev's polynom expansion.

Later Rasmussen and Smith [6] developed an interesting method based on the rigorous analysis of Fourier's series. It was shown their method converges faster than a collocation manner having the same number of equations. However this method was devised only for symmetrical and smooth incidence distribution over the span.

It should be noted, in spite of all sorts of developments, mentioned above analytical methods were rather complicated and lengthy.

As for numerical methods the Yuriev's [7] and J. Anderson's [8] works may be named where they used iteration approach with consecutive improvement of tentatively set circulation. Results slowly converged to constant magnitudes. However it was noted [7] that in comparison with the analytical methods numerical ones may be used also for wings with: complicated plan-forms, non-smooth incidence distribution along the span and non-linear airfoil characteristics.

## III. TASK STATEMENT

The goal of this work was to present a noniterative numerical solution method of the Prandtl integro-differential equation, that had a simple algorithm and would enable to compute fast a valid enough span-wise circulation distribution, which may be used later for predicting aerodynamic characteristics of finite span wings.

## IV. DESCRIPTION OF METHOD

The Prandtl singular integro-differential equation for finite span wings [2] was written in a form:

$$\Gamma(z)=\frac{1}{2}C_y^{\alpha}bV\left[\alpha+\frac{1}{4\pi V}\int_{-l/2}^{l/2}\frac{d\Gamma(\zeta)}{d\zeta}\frac{d\zeta}{\zeta-z}\right], \quad (1)$$

where $\Gamma(z)$ is the span-wise circulation distribution;

$z$ is the coordinate along the span;

$C_y^{\alpha}$ is the airfoil lift slope;

$b$ is the chord length;

$V$ is the incoming flows velocity;

$\alpha$ is the angle of incidence counted from an aerodynamic chord;

$l$ is the wing span;

$\zeta$ is the coordinate for integrating along the span.

Generally speaking magnitudes of $C_y^{\alpha}, b, V, \alpha$ may be variable over the span.

Boundary conditions [2]:

$$\Gamma\left(\pm\frac{l}{2}\right)=0. \quad (2)$$

Equation (1) was rewritten in a form of a singular integral one:

$$\int_{-l/2}^{z}\gamma(\zeta)d\zeta=\frac{1}{2}C_y^{\alpha}bV\left[\alpha+\frac{1}{4\pi V}\int_{-l/2}^{l/2}\frac{\gamma(\zeta)d\zeta}{\zeta-z}\right], \quad (3)$$

where $\gamma(\zeta)=\Gamma'(\zeta)$ is the vortex intensity at the point $\zeta$.

So instead of the source problem (1) we would solve the one (3) with boundary conditions (2).

## V. COMPUTATIONAL ALGORITHM OF METHOD

We implemented a discretization of the problem (3). A wing was divided span-wise into $n$ elements. In the case of uniform distribution an elementary span was:

$$\Delta z=\frac{l}{n}.$$

An assumption was made that a vortex intensity $\gamma_j$ to be predicted was a constant one within each of elements and located in an internal point between the bounds. Coordinates $\zeta_j$ indicated internal points and coordinates $z_m$ indicated elements bounds. It enabled to avoid the singularity in the discrete model.

Thus equation (3) may be written for every element. Span-wise integration was approximately replaced with summation over the discrete elements of the wing:

$$\sum_{j=1}^{m}\gamma_j\Delta z_j=\frac{1}{2}C_{ym}^{\alpha}b_mV_m\left[\alpha_m+\frac{1}{4\pi V_m}\sum_{j=1}^{n}\frac{\gamma_j\Delta z_j}{\zeta_j-z_m}\right], \quad (4)$$

where an index $m$ meant, that appropriate magnitude was related to the section $z_m$.

Equation (3) with the boundary conditions (2) was brought to the linear algebraic equations system (4) with the unknown intensities $\gamma_j$. The number of equations equaled the number of elements.

After the solution of the algebraic equations system the span-wise circulation distribution $\Gamma(z)$ may approximately be calculated by the left side expression of the formula (4). It would enable to calculate lift, induced drag and to evaluate pitching moment of the wing.

## VI. THE WAY OF POINTS SELECTION

Calculations of span-wise circulation distribution for wings with elliptical, trapezoidal (with taper ratio $\eta=0.5$), rectangular and triangular plan-forms were performed for approbation and reliability determining of the method. The wings had following parameters: span $l=2$ m; aspect ratio $\lambda=5.6$; lift curve slope $C_y^{\alpha}=5.6\ \text{rad}^{-1}$. An angle of incidence $\alpha=1$ rad was accepted.

Figures 1 – 3 presented results of comparative calculations with the basic Glauert's analytical method. Graphics of the latter were marked with thick solid lines and computed for a collocation points number $n$ = 8.

Results of calculations for the wing with a rectangular plan-form on the uniform mesh over the span were set on Fig.1. Internal points located in the middles of the elements. So in comparison with the analytical method this one undervalued entirely the span-wise circulation especially in the vicinity of the wing tips. Increase of number elements reduced undervalueing only. It suggested itself elements crowding at wing tips.

It was known that Glauert's collocation points were set through angle coordinates with formula [2]:

$$z_j=-\frac{l}{2}\cos\theta_j, \quad (5)$$

where $\theta_j$ is the angles obtained by means of uniform dividing from 90 to 180 degrees.

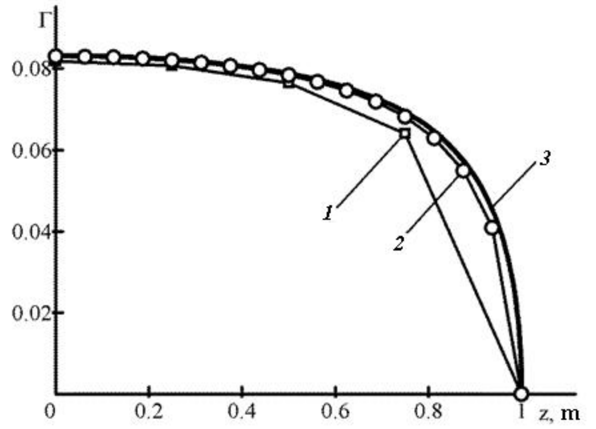


Fig. 1. Circulation distribution for the rectangular wing on the uniform mesh: *1* is the number of sections $n = 4$; *2* is the number of sections $n = 16$; *3* is the Glauert's method

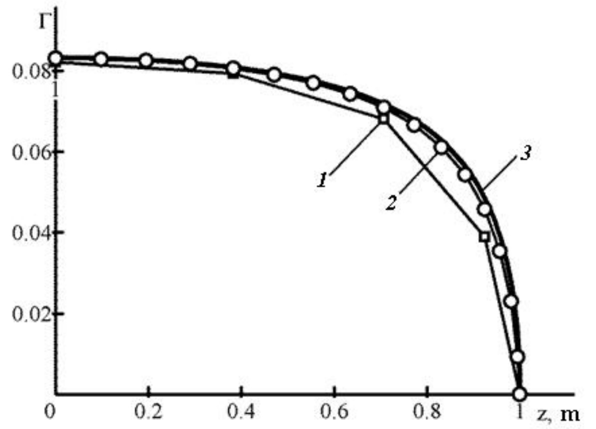


Fig. 2. Circulation distribution for the rectangular wing on the non-uniform mesh: 1 is the number of sections $n = 4$; 2 is the number of sections $n = 16$; 3 is the Glauert's method

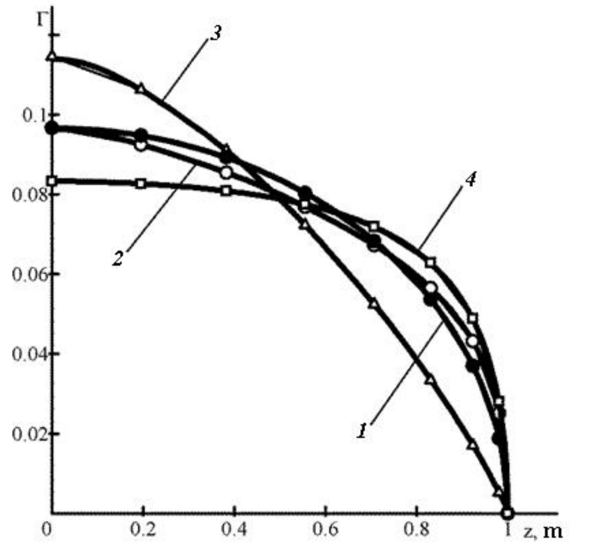


Fig. 3. Circulation distribution on the completely non-uniform mesh with $n = 8$ for the wing plan-forms: *1* is the elliptical; *2* is the trapezoid; *3* is the triangular; *4* is the rectangular

Results of calculations for the same wing on the non-uniform mesh over the span were set on the Fig. 2. Here the coordinates of elements bounds were determined by the formula (5) through uniform distributed angles. Again internal points located in the middles of the elements. It may be asserted that convergence of the two methods was better. But again increase of elements number reduced undervalueing only.

The results for the four plan-forms on the non-uniform mesh over the span were set on Fig. 3. All the points were determined here by the eq. (5). The full convergence of results was observed.

## CONCLUSIONS

The rewriting of the source Prandtl singular integro-differential equation in a form of singular integral one in combination with the special setting of the non-uniform mesh through uniform distributed angle coordinates enabled to obtain the spanwise circulation distribution without iterations.

Generally speaking the results comparisons indicate the presented numerical method may be used as an alternative one for calculating a circulation distribution of a finite span wing.

Since the method turned out to be the simple and the reliable one it may be employed as algorithm for CAD systems.

It is recommended to test the method for wings with: complicated plan-forms, non-smooth incidence distribution along the span, non-linear airfoil characteristics and also for nonsymmetrical, non-uniform and unsteady flows.

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Received 03 April 2015.

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**А. А. Зіганшин. Числове розв'язання інтегро-диференційного рівняння крила**

Представлено метод числового розв'язання інтегро-диференційного рівняння Прандтля для крила. Проведено порівняльні розрахунки сумісно з аналітичним методом Глауерта. Отримано гарний збіг результатів на нерівномірній сітці.

**Ключові слова:** аеродинаміка; вихор; крило; теорія несучої лінії; циркуляція.

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Напрям наукової діяльності: системи автоматизації проектувальних робіт, числові методи в аеродинаміці.
Кількість публікацій: 2.
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**А. А. Зиганшин. Численное решение интегро-дифференциального уравнения крыла**

Представлен метод численного решения интегро-дифференциального уравнения Прандтля для крыла. Проведены сравнительные расчеты совместно с аналитическим методом Глауэрта. Получено хорошее совпадение результатов на неравномерной сетке.

**Ключевые слова:** аэродинамика, вихрь; крыло; теория несущей линии; циркуляция.

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