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User's data positioning by GPS and GLONASS navigation satellites measurements

The calculation algorithm of user's position data is cited with the addition of GPS and GLONASS navigation satellites measurements. The algorithm is generalized on certain types of satellite-based navigation systems. The generalized algorithm is illustrated by the results of the full-scale modeling experiment.

The pseudo-range method is more common in calculation of the position data of a navigation receiver.

The method is based on the equation

$$PR_i^{meas} = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} + h_n + \Delta_i , \qquad (1)$$

where

x, y, z- coordinates subject to definition,

– navigation satellite coordinates with number *i*,

PR^{meas} - the measured pseudo-range of the navigation satellite with number *i*,

h_n - a pseudo-range telemetry error due to the discrepancy of time scales of the satellite and the receiver,

Δ_i- inaccuracy of pseudo-range positioning of an i-satellite because of ephemerid prediction errors,   errors   of   time-and-frequency   maintenance,   errors   of radio-wave velocity in troposphere and ionosphere on "i-navigation satellite-service" routes, errors of multipath transmission of satellite signals in the place of reception, noise of the reception channel of user's equipment and errors of natural and deliberate interference.

In equation (1) there are four unknown elements subject to definition: *x, y, z, h_n.*. One should obtain pseudo-range survey for at least four satellites to determine *x, y, z* data and an h_n error. In a one-stage survey h_n is identical, so a correct solution can be made in four nonlinear equations (1).

If the navigation receiver measures the pseudo-range according to the GPS and GLONASS systems operating in different time scale systems, an unfamiliar value is added – an error of pseudo-range survey due to the discrepancy of GLONASS time scale satellite and (h_k) receiver. For determining *x, y, z* elements and h_k error the equation (1) becomes as follows:

$$PR_j^{meas} = \sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2} + h_k + \Delta_j , \qquad (2)$$

where

x, y, z-coordinates subject to definition,

x_j, y_j, z_j- GLONASS navigation satellite coordinates with number *j*,

PR_j^{meas}- the measured pseudo-range of a GLONASS navigation satellite with number j,

h_k - pseudo-range telemetry error due to the discrepancy of time scales of a GLONASS satellite and the receiver,

Δ_j- inaccuracy of pseudo-range positioning of an i-satellite because of ephemerid prediction errors,   errors   of   time-and-frequency   maintenance,   errors   of radio-wave velocity in troposphere and ionosphere on "i-navigation satellite-service" routes,

errors of multipath transmission of satellite signals in the place of reception, noise of the reception channel of user's equipment and errors of natural and deliberate interference.

Thus, to determine user's data with the addition of pseudo-range surveys of GPS and GLONASS satellites one should find out five unknown elements: *x, y, z, h_n, h_k.* In doing so there are at least five equations as (1,2). And they should concerns to various satellite systems of any quantitative combination, for example: 4 GPS + 1 GLONASS, 3 GPS + 2GLONASS, 2 GPS + 3 GLONASS, 1 GPS + 4 GLONASS.

Since there are 6 to 11 GPS and GLONASS satellites (with GLONASS full expansion) in the visibility area of the navigation receiver, it is expedient to determine user's data through pseudo-range surveys for all visible satellites. As proved experimentally and theoretically, the larger the number of measured pseudo-ranges is used in data positioning, the more accurate and reliable the calculation results are.

If the number of measured pseudo-range is more than 5, iterative algorithms are advisable to determine user's position data. The main point of the iterative algorithm is the following:

1. The equations system connecting measured and design pseudo-range for every navigation satellite from equations (1,2) is established.
2. Linearization of the equations system is provided through Taylor series factorization with linear term deduction only.
3. The least-squares method is applied to the linearization system for optimum solution of the redefined equation system (the number of the equations is more than the unknown quality).
4. User's data are determined under the recurrent formula following from item 3.

Let's sum up necessary correlation for the above- said algorithm.

Suppose we have two groups of satellites: GPS satellites with 1 to n (n-group) numbers and GLONASS satellites with $n+1$ to k (k-group) numbers.

$n+k$ is bigger than 5 and n and k groups require pseudo-range measurement of at least one satellite.

Let's write down the equations (1, 2) for groups of n and k satellites with reference to the design pseudo-range:

$$
\begin{aligned}
PR_i^{calc} &= \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} + h_n + 0 \cdot h_k, 1 \le i \le n; \\
PR_j^{calc} &= \sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2} + 0 \cdot h_n + h_k, n+1 \le j \le n+k.
\end{aligned}
\tag{3}
$$

Then we add a vector of the parameters subject to definition

$$[X] = [x, y, z, h_n, h_k]^T,$$

where "T" – means transposion.

Then we'll differentiate (3) on vector $[X]$ and will write the obtained result as a matrix, having marked it as $[G([X])]$

$$\left[\frac{\partial}{\partial [X]}PR^{calc.}\right]=\begin{bmatrix} \dfrac{-(x_1-x)}{D_1} & \dfrac{-(y_1-y)}{D_1} & \dfrac{-(z_1-z)}{D_1} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{-(x_n-x)}{D_n} & \dfrac{-(y_n-y)}{D_n} & \dfrac{-(z_n-z)}{D_n} & 1 & 0 \\ \dfrac{-(x_{n+1}-x)}{D_{n+1}} & \dfrac{-(y_{n+1}-y)}{D_{n+1}} & \dfrac{-(z_{n+1}-z)}{D_{n+1}} & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{-(x_{n+k}-x)}{D_{n+k}} & \dfrac{-(y_{n+k}-y)}{D_{n+k}} & \dfrac{-(z_{n+k}-z)}{D_{n+k}} & 0 & 1 \end{bmatrix}=[G([X])]. \qquad (4)$$

Now we shall explain matrix record (4). If we have, for example, five equations with five unknown elements x_1, x_2, x_3, x_4, x_5, they can be written as follows:

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{55}x_5 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5 \\ y_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + a_{45}x_5 \\ y_5 &= a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + a_{55}x_5 \end{aligned}\right\}. \qquad (5)$$

In a matrix way equation system (5) is written like this:

$$[y]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{53} & a_{53} & a_{54} & a_{55} \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \qquad (6)$$

Thus, equation system (5) requires the following
$$a_{14} = a_{24} = a_{34} = a_{45} = a_{55} = 1,$$
$$a_{15} = a_{25} = a_{35} = a_{44} = a_{54} = 0.$$

Then system (5) can be written as

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + 1\cdot x_4 + 0\cdot x_5 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + 1\cdot x_4 + 0\cdot x_5 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + 1\cdot x_4 + 0\cdot x_5 \\ y_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + 0\cdot x_4 + 1\cdot x_5 \\ y_5 &= a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + 0\cdot x_4 + 1\cdot x_5 \end{aligned}\right\} \qquad (5,\ a)$$

or in matrix representation

$$[y]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 1 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 1 \\ a_{51} & a_{53} & a_{53} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \tag{6, a}$$

Expression (4) shows that hn and h_k are met only in their own equation groups as GPS and GLONASS time scale systems are independent: *0* indicates absence of hn derivative, in a *k* group (or h_k in an n group), 1 – availability of the derivatives.

Let's enter function from a vector *[X]* as a difference between measured and design pseudo-ranges

$$R_i([X]) = PR_i^{\text{meas.}} - PR_i^{\text{calc.}}, 1 \le i \le n+k. \tag{7}$$

Function (7) is Taylor series expansion for all *i* lines with linear term deduction close to vector $[X_0] = [x_0, y_0, z_0, h_{n0}, h_{k0}]^T$

$$R_i([X]) = PR_i^{\text{meas.}} - PR_i^{\text{calc.}}([X_0]) + \frac{\partial}{\partial([X])} R_i([X])_{[X]=\{X_0\}} \cdot ([X]-[X_0]). \tag{8}$$

In matrix representation expression (8) is written as

$$[R([X])] = [PR^{\text{meas}}] - [PR^{\text{calc}}([X_0])] - [G_0] \cdot [X] + [G_0] \cdot [X_0], \tag{9}$$

where *[G₀]* is matrix (4) in its substitution of vector *[X₀]*.

To make it general we shall put a weight factor matrix in equation system (9), having multiplied both left parts of the system by a diagonal matrix *[P]*, whose factors P_I are also weight factors

$$[P] \cdot [R([X])] = [P] \cdot \{[PR^{\text{meas}}] - [PR^{\text{calc}}([X_0])] - [G_0] \cdot [X] + [G_0] \cdot [X_0]\}. \tag{10}$$

Required parameters-data of the user are in the system's vector *[X]* of equations (10). As the equations system (10) is redefined, it can be optimally solved by the least-squares method. In doing so the left part of system (10) is treated as position error (residual). Then the quadratic form of expression (10) is written as follows

$$[\varepsilon]^T \cdot [\varepsilon] = \{[\Psi] - [G_0] \cdot [X]\}^T \cdot [P]^T \cdot [P] \cdot \{[\Psi] - [G_0] \cdot [X]\}, \tag{11}$$

where

$[\varepsilon] = [P] \cdot [R([X])],$

$[\Psi] = [P] \cdot \{[PR^{\text{meas}}] - [PR^{\text{calc}}([X_0])] + [G_0] \cdot [X_0]\}.$

By differentiating (11) vector *[X]* and equating the result with a zero vector we receive

$$[G_0]^T [P]^T [P] \cdot \{[PR^{\text{meas}}] - [PR^{\text{calc}}([X_0])] + [G_0] \cdot [X_0] - [G_0] \cdot [X]\} = [0]. \tag{12}$$

The following comes from expression (12)

$$[X] = [X_0] + \{[G_0]^T \cdot [P]^T \cdot [P] \cdot [G_0]\}^{-1} \cdot [G_0]^T \cdot [P]^T \cdot [P] \times$$
$$\times \{[PR^{meas}] - [PR^{calc}([X_0])]\}.$$

(13)

To make expression (13) clear we shall transfer it into a recurrent formula and make certain substitutions:

$$[P]^T \cdot [P] = [W]^{-1},$$
$$[PR^{meas}] = [PR^{(m+1)}],$$
$$[PR^{calc}([X_0])] = [PR^{(m)}],$$
$$[G_0] = [G^{(m)}]$$

$$[X] = [X^{(m+1)}],$$
$$[X_0] = [X^{(m)}].$$

In view of the substitutions made the recurrent algorithm for determining use's position data by GPS

and GLONASS satellite measurements will be written as follows

$$[X^{(m+1)}] = [X^{(m)}] + \{[G^{(m)}]^T \cdot [W]^{-1} \cdot [G^{(m)}]\}^{-1} \cdot [G^{(m)}]^T \cdot [W]^{-1} \cdot \{[PR^{(m+1)}] - [PR^{(m)}]\}. \quad (14)$$

Algorithm (14) is quick convergent, yet matrix *[W]* is individual for standard measurements, all calculations on algorithm are carried out in one system of coordinates (P390 or WGS-84). For the first inclusion of the algorithm a zero vector can be accepted as the basic one, but during continuous stationary measurements the initial vector can be the result of the previous calculation.

The stated algorithm can be generalized as rho-rho sources belonging to several systems rather than two ones, for example, GPS, GLONASS, a geostationary satellite of the EGNOS, GALILEO system, pseudo-satellites, stations of LORAN-C system etc.

The generalized algorithm is similar to expression (14), matrix *[G]* is formed in the following way. Suppose we have measurements for rho-rho sources of systems S_i, S_2…S_n. Then matrix *[G]* will look like:

$$[\tilde{G}] = \begin{pmatrix} \dfrac{-(x_{s1}-x)}{D_{s1}} & \dfrac{-(y_{s1}-y)}{D_{s1}} & \dfrac{-(z_{s1}-z)}{D_{s1}} & 1_{s1} & 0_{s1} & ... & 0_{s1} \\ (.)_{s1} & (..)_{s1} & (...)_{s1} & 1_{s1} & 0_{s1} & ... & 0_{s1} \\ \dfrac{-(x_{s2}-x)}{D_{s2}} & \dfrac{-(y_{s2}-y)}{D_{s2}} & \dfrac{-(z_{s2}-z)}{D_{s2}} & 0_{s2} & 1_{s2} & ... & 0_{s2} \\ (.)_{s2} & (..)_{s2} & (...)_{s2} & 0_{s2} & 1_{s2} & ... & 0_{s2} \\ ... & ... & ... & ... & ... & ... & ... \\ \dfrac{-(x_{sn}-x)}{D_{sn}} & \dfrac{-(y_{sn}-y)}{D_{sn}} & \dfrac{-(z_{sn}-z)}{D_{sn}} & 0_{sn} & 0_{sn} & ... & 1_{sn} \end{pmatrix}.$$

(15)

Sequential numeration of lines and columns is used in matrix (15) for parameters of rho-rho sources of systems S_i, $S_2...S_N$ respectively. Thus first three columns of matrix (15) include data of all rho-rho sources and data to be defined (x, y, z). Factors D as well as in (4) are figured out from the formula:

$$D_{(.)} = \sqrt{(x_{(.)} - x)^2 + (y_{(.)} - y)^2 + (z_{(.)} - z)^2} .$$

For the stipulated numeration columns in (15) starting with the fourth contain zeros and units. And units should be on crosstracking with rho-rho sources data belonging to one navigation system and the correspondent column. In all other cases zeros are written in the columns beginning with the fourth. Each line may contain only one unit.

To illustrate the suggested algorithms we shall cite some calculation results of stationary point data obtained from GPS, GLONASS rho-rho sources and geostationary satellite.

Data of stationary points (meters):

Rx=3504451.023; Ry=2061316.876; Rz=4897990.975.

Data of GLONASS satellites (each line corresponds to coordinates *X, Y, Z* of the corresponding satellite, (hereinafter dimension is made in meters):

```
1.392092871100000e+007    1.676323046900000e+007    1.319036914100000e+007
2.188252441000000e+006    2.393134570300000e+007    8.537842773000000e+006
9.199756835999999e+006    1.071107031200000e+007    2.120916210900000e+007
1.084207421900000e+007   -9.068601562000001e+006    2.122262109400000e+007.
```

Measured pseudo-range for GLONASS satellites:
```
1.9700236328e+007    2.2075472656e+007    1.9186814453e+007    2.0942015625e+007.
```
GPS satellite data (each line corresponds to coordinates *X, Y, Z* of the corresponding satellite)

```
 6.679790500000000e+006   -1.511358300000000e+007    2.155421600000000e+007
 8.013218500000000e+006    1.924829200000000e+007    1.654482900000000e+007
-9.249662000000000e+006    1.154506200000000e+007    2.201939800000000e+007
 2.517969600000000e+007    5.563761250000000e+005    8.465181000000000e+006
 1.313889700000000e+007   -8.312081000000000e+006    2.159311400000000e+007
 1.823084600000000e+007    8.988665000000000e+006    1.755615800000000e+007
 2.062122800000000e+007    1.584874400000000e+007   -5.799892000000000e+006
 1.848023800000000e+007    1.911264000000000e+007   -7.624061880000000e+005
 2.478954400000000e+007   -9.403282000000000e+006    1.018964300000000e+005
 1.729687200000000e+007    1.980525000000000e+007   -4.271833000000000e+006.
```

Measured pseudo-range for GPS satellites:

24000304 21110926 23226778 21883804 21755152 20483034 24309728 23254842 24512886 24138180.

GEO satellite data and pseudo-range:

X = 18483400; *Y* = 37897600; *Z* = 0.0; псевдодальность = 3.914841524493983e+007.

Data errors of the stationary point determined according to the mentioned algorithms and geometrical factors for typical configurations of satellite systems are illustrated in the table.

#	Configuration	ΔX	ΔY	ΔZ	PDOP	HDOP
1	4 satellites GLONASS	-2.87e-01	1.319e+02	7.31e+01	4.18	3.02
2	4 satellites GPS (PRN: 2, 6, 10, 15)	3.62e-01	1.896e+01	5.459e+01	7.68	3.18
3	4 satellites GPS (PRN: 16, 17, 18, 23)	2.041e+02	3.651e+01	1.109e+01	10.9	10.73
4	4 satellites GPS (PRN: 10, 15, 16, 17)	-5.32	4.038e+01	2.157e+01	3.64	2.13
5	4 satellites GPS (PRN: 2, 6, 25, 30)	5.708e+01	1.559e+01	5.065e+01	5.42	4.57
6	4 satellites GPS (PRN: 2, 10, 17, 30)	6.31	16.04	25.33	2.26	1.66
7	5 satellites GPS (PRN: 2, 10, 17, 25, 30)	-4.78	5.98	13.86	1.95	1.47
8	6 satellites GPS (PRN:2, 10, 17, 18, 25, 30)	-5.43	7.79	8.45	1.88	1.48
9	7 satellites GPS (PRN:2, 10, 15, 16, 17, 18, 25)	-19.4	14.74	-9.74	1.77	1.35
10	8 satellites GPS (PRN:2, 10, 15, 16, 17, 18, 25. 30)	-16.56	9.54	2.32	1.7	1.32
11	10 satellites GPS (PRN:2, 6, 10, 15, 16, 17, 18, 23, 25. 30)	-15.75	10.5	2.95	1.55	1.22
12	10 satellites GPS + 4 ГЛОНАСС	-12.07	42.28	9.01	1.36	1.06
13	8 satellites GPS (PRN:2, 10, 17, 18, 25, 30) + 1 GLONASS (PRN: 55)	-5.42	7.78	8.45	2.72	1.88
14	10 satellites GPS + 4 GLONASS + 1 GEO	-12.07	42.27	9.01	1.37	1.06
15	10 satellites GPS + 3 GLONASS + 1 GEO	-17.86	10.38	0.23	1.41	1.12

The data resulted in the table are not a subject for the analysis of various constellations of navigation satellites. This issue needs individual consideration. The data in the table illustrate opportunities of above-mentioned algorithms and are also test examples for experts designing software for navigation receivers.