MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National aviation university

HIGHER MATHEMATICS

NUMBER AND FUNCTIONAL SERIES

A guide
to independent work of higher education applicants
of technical specialties

Kyiv 2022

UDC

Compilers:
I.O. Lastivka – Doctor of Engineering, Professor, Head of the Department of Higher Mathematics

Y.G.Liashenko – PhD in Physics and Mathematics, Associate Professor

Reviewer: PhD in Pedagogy, Associate Professor V.I.Trofymenko

Approved by the Methodological and Editorial Board of the National Aviation University (Minutes № ___ of ________ 2022).

Higher mathematics. Number and functional series: A guide to independent work of higher education applicants of technical specialties/compil.: I.O. Lastivka, Y.G.Liashenko – К. : NAU, 2022. –48p.

The guide is compiled according to the syllabus of the "Higher Mathematics" subject. The guide includes examples of solutions of typical problems of the topic "Number and functional series", questions and tasks for self-test and problems for self-assignments.

For higher education applicants of technical specialties

CONTENTS

INTRODUCTION

Independent work of the higher education applicant is the main way of mastering the educational material during the time free from the compulsory classroom classes.

The purpose of independent work - deepening, generalization and consolidation of theoretical knowledge and practical skills of students in the "Higher Mathematics" subject by developing the ability to work independently with the academic literature.

Independent work of higher education applicants is carried out in the form of preparation for lectures and practical classes, performance of individual homework and performance of module tests. Such training involves independent study of theoretical material on each topic presented in the recommended literature and lecture notes. It is important to pay attention to the need for a clear assimilation of basic terms and definitions, understanding of their content, obligatory analysis of the use of theoretical information for the proposed tasks.

The guide for independent work of higher-education applicants is compiled in accordance with the curricula of the course "Higher mathematics" for students of technical specialties. The proposed methodical work presents tasks for independent and individual work. A significant number of tasks for independent work has an applied orientation.

The leading teacher can adjust the number and content of the tasks, which a student must perform independently while studying the relevant material.

The material of each topic corresponds to the working curricula of the "Higher mathematics" subject, in particular, to one of its sections "Number series". Each topic contains the basic methodical recommendations, recommended literature, typical examples of solutions and tasks for individual performance, and questions for self-checking, which will contribute to better understanding, assimilation and possibility to apply the basic theoretical statements.

This guide is compiled for independent work of higher education applicants of technical specialties and focused on the theoretical and methodological support of the training process.

Topic 1. NUMBER SERIES

Plan

1. The main concepts and definitions, convergence.
2. Properties of number series.
3. The necessary condition of convergence. The sufficient condition of divergence.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 1 the student should ***know***: definition of numerical series, partial sum of series, convergence of series, sum of series, examples of known convergent and divergent series, necessary condition of convergence and sufficient condition of divergence; ***be able to***: calculate the sum of convergent number series of a certain types.

Basic theoretical information

Let $\{u_n\} = \{u_1, u_2, \ldots, u_n, \ldots\}$ be the sequence of real numbers.
Expression

$$u_1 + u_2 + ... + u_n + ... = \sum_{n=1}^{\infty} u_n \tag{1.1}$$

is called *number series* (or series)

Here u_1 is the first, u_2 is the second, ..., u_n is n-th (general) term of series. The series (1.1) is given if the dependence of its general term on the number *n* is known: $u_n = f(n)$.

The sum $S_n = u_1 + u_2 + \ldots + u_n$ of the first *n* terms of the series is called *n-th partial* sum of series (1.1).

If there exists the finite limit $\lim\limits_{n\to\infty} S_n = S$ then series (1.1) is called *convergent* and number S is called *sum* of this series.

If limit $\lim\limits_{n\to\infty} S_n$ doesn't exist or $\lim\limits_{n\to\infty} S_n = \infty$ then series (1.1) is called *divergent.* Divergent series has no sum.

Expression $r_n = S - S_n = u_{n+1} + u_{n+2} + ...$ is called *n-th* remainder of series.

Properties of number series

1. If the series $\sum_{n=1}^{\infty} u_n$ is convergent and its sum is equal to S then the series $\sum_{n=1}^{\infty}(Cu_n)$ (where C is constant) is convergent as well. And its sum is equal to the product CS. If the series $\sum_{n=1}^{\infty} u_n$ diverges and $C \neq 0$ then series $\sum_{n=1}^{\infty}(Cu_n)$ also diverges.

2. If number series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, moreover S_u and S_v are their sums correspondently then series $\sum_{n=1}^{\infty}(u_n + v_n)$ and $\sum_{n=1}^{\infty}(u_n - v_n)$ are also convergent and their sums are equal $S_u + S_v$ and $S_u - S_v$ respectively.

Remark. The sum (difference) of convergent and divergent series is a divergent series. The sum (difference) of two divergent series can be either a convergent or a divergent series.

3. The convergence of a series does not depend on the discard or addition of a finite number of terms.

Theorem 1.1 (*necessary condition for convergence of a series*). If series (1.1) converges then its general term u_n tends to zero, i.e. $\lim_{n\to\infty} u_n = 0$.

Conclusion. In order to a series converges its general term must tends to zero. However, it is not a guarantee of the convergence of the series. If $\lim_{n\to\infty} u_n = 0$ then it only means that the series $\sum_{n=1}^{\infty} u_n$ can be convergent.

Corollary *(sufficient condition of divergence of series).* If $\lim_{n\to\infty} u_n \neq 0$ or this limit does not exist, then series (1.1) diverges.

Examples of solution of typical problems

Example 1. Write down the general, the first and the third terms of the series $\sum_{n=1}^{\infty}\frac{n+2}{n^2+4}$.

Solution.

General (nth) term of series is $u_n=\frac{n+2}{n^2+4}$. Then $u_1=\frac{1+2}{1+4}=\frac{3}{5}$ is the first term of series; $u_3=\frac{3+2}{9+4}=\frac{5}{13}$ is the third term of series.

Example 2. Investigate the series for convergence. If the series converges, calculate its sum:

a) $1+4+7+\ldots+(3n-2)+\ldots=\sum_{n=1}^{\infty}(3n-2)$;

b) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}$;

c) $b_1+b_1q+b_1q^2+\ldots+b_1q^{n-1}+\ldots=\sum_{n=1}^{\infty}b_1q^{n-1}$;

d) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n}$.

Solution:

a) let's write n-th partial sum

$S_n=1+4+7+\ldots+(3n-2)+\ldots=\frac{1+(3n-2)}{2}n=\frac{(3n-1)n}{2}$. Here the formula $S_n=\frac{a_1+a_n}{2}n$ for the sum of the first n terms of arithmetic progression is used. Hence, $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{(3n-1)n}{2}=\infty$ and we can say that the given series is divergent;

b) we write down and transform partial sum S_n of this series in following way:

$$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+$$

$+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$. Since $\lim\limits_{n\to\infty} S_n=\lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1,$ then the series is convergent and its sum $S=1$.

Remark. If $u_1=f(1)-f(2),$ $u_2=f(2)-f(3),\ldots,$ $u_n=f(n)-f(n+1),$ then $u_1+u_2+\ldots+u_{n-1}+u_n=f(1)-f(n+1).$

c) the series $b_1+b_1q+b_1q^2+\ldots+b_1q^{n-1}+\ldots=\sum\limits_{n=1}^{\infty}b_1q^{n-1}$, $(b_1\neq 0)$ is called a series of geometric progression (or *geometric series*): if $q=1$ then this series is divergent; if $q\neq 1$:

$$S_n=b_1+b_1q+b_1q^2+\ldots+b_1q^{n-1}=\frac{b_1(1-q^n)}{1-q}=\frac{b_1}{1-q}-\frac{b_1q^n}{1-q}.$$

Let's find limit of this sum depending on value q:

1) if $|q|<1$ then $\lim\limits_{n\to\infty}q^n=0$, and then $\lim\limits_{n\to\infty}S_n=\frac{b_1}{1-q}$. Hence, the series is convergent and its sum $S=\frac{b_1}{1-q}$;

2) if $|q|>1$ then $\lim\limits_{n\to\infty}q^n=\infty$, and then $\lim\limits_{n\to\infty}S_n=\infty$. Hence. The series is divergent;

3) if $q=-1$ then the series acquires the form:

$b_1-b_1+b_1-b_1+\ldots+(-1)^{n+1}b_1+\ldots$ and it is divergent.

Conclusion. The geometric series $\sum\limits_{n=1}^{\infty}b_1q^{n-1}$ $(a\neq 0)$ converges for $|q|<1$ and diverges for $|q|\geq 1$.

d) the series $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots=\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is called *harmonic*. This series is divergent.

Example 3. Find the limit of general term u_n of series $\sum\limits_{n=1}^{\infty}u_n$ and make a conclusion according to its convergence:

a) $\sum_{n=1}^{\infty}\frac{2n}{11n+1}$; b) $\sum_{n=1}^{\infty}2^{-n}$; c) $\sum_{n=1}^{\infty}\cos\frac{1}{n}$; d) $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^{n}$.

Solution:

a) Since $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{2n}{11n+1}=\frac{2}{11}\neq 0$ then given series is divergent;

b) $\sum_{n=1}^{\infty}2^{-n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$ is convergent geometric series ($q=\frac{1}{2}<1$). For this series $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^{n}=0$;

c) $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\cos\frac{1}{n}=\cos 0=1\neq 0$. Hence, this series is divergent;

d) $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left(\frac{n-2}{n}\right)^{n}=\left(1^{\infty}\right)=\lim\limits_{n\to\infty}\left(1-\frac{2}{n}\right)^{n}=e^{-2}\neq 0$. Hence, the series is divergent;

Self-test questions

1. What is called a numerical series?
2. How to determine the first, sixth, tenth terms of the series?
3. What is the *n*-th partial sum of a series?
4. Formulate the definition of a convergent series and its sum.
5. What can you say about the convergence of a series $\sum_{n=1}^{\infty}u_n$ if $\lim\limits_{n\to\infty}u_n=0$?
6. Give examples of convergent and divergent series.
7. Is the statement correct: if a series $\sum_{n=1}^{\infty}u_n$ converges, then the series $\sum_{n=1}^{\infty}\left(u_n+\frac{1}{n}\right)$ also converges?

Self-test assignments

Task 1. Prove the convergence of the series by the definition and find its sum:

a) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$; b) $\sum_{n=1}^{\infty}\frac{2}{n(n+2)}$; c) $\sum_{n=1}^{\infty}\frac{2^n+3^n}{6^n}$;

d) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$; e) $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$; f) $\sum_{n=1}^{\infty}\sin\frac{\pi}{2^{n+2}}\cos\frac{3\pi}{2^{n+2}}$.

Task 2. Prove the divergence of the series using sufficient condition of divergence of series:

a) $\sum_{n=1}^{\infty}\left(\sqrt{n^2+4n}-n\right)$; b) $\sum_{n=1}^{\infty}\frac{5n-1}{1000n+1}$; c) $\sum_{n=1}^{\infty}\left(\frac{n-2}{n+3}\right)^n$;

d) $\sum_{n=1}^{\infty}\frac{n}{\ln(n+1)}$; e) $\sum_{n=1}^{\infty}\cos\frac{1}{n^2}$; f) $\sum_{n=1}^{\infty}\sin\frac{\pi n}{2n+1}$.

Answers: 1. a) $S_n=\frac{1}{2}-\frac{1}{4n+2}$, $S=\frac{1}{2}$; b) $S_n=\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}$, $S=\frac{3}{2}$; c) $S_n=\frac{3}{2}-\frac{1}{2^n}-\frac{1}{2\cdot 3^n}$, $S=\frac{3}{2}$; d) $S_n=1-\frac{1}{(n+1)^2}$, $S=1$. *Instruction.* $\frac{2n+1}{n^2(n+1)^2}=\frac{(n+1)^2-n^2}{n^2(n+1)^2}=\frac{1}{n^2}-\frac{1}{(n+1)^2}$; e) $S_n=1-\frac{1}{(n+1)!}$, $S=1$. *Instruction* $\frac{n}{(n+1)!}=\frac{1}{n!}-\frac{1}{(n+1)!}$; f) $S_n=\sin\frac{\pi}{2}-\sin\frac{\pi}{2^{n+1}}$, $S=1$. *Instruction*. Use the formula $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$.

Topic 2. TESTS FOR CONVERGENCE OF POSITIVE TERMS SERIES

Plan

1. Definition of positive terms series, examples of reference series.
2. Tests for convergence of positive terms series.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 2 the student should ***know***: definition of positive terms numerical series, examples of known reference series, comparison tests, D'Alembert's test, Cauchy's test, integral test; ***be able to***: examine the positive terms series for convergence by the corresponding tests.

The basic theoretical information

Series with non-negative terms are called *positive terms series*. We will use comparison tests, D'Alembert's test, Cauchy's test and an integral test to examine the convergence of such series.

The convergence or divergence of positive terms series is sometimes established by comparing it with a series whose behavior is known. Such series are called *reference* series.

The following reference series are most often used:

a) geometric series;

b) harmonic series;

c) generalized harmonic series (or *Dirichlet-Riemann* series):

$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+...$, which is convergence for $p>1$ and divergent for $p\le 1$.

Theorem 2.1 (*Comparison test*). Let

$$\sum_{n=1}^{\infty}u_n=u_1+u_2+\ldots+u_n+\ldots, \tag{2.1}$$

$$\sum_{n=1}^{\infty}v_n=v_1+v_2+\ldots+v_n+\ldots, \tag{2.2}$$

be two series of positive terms, so that: $0\le u_n\le v_n$ ($n=1,2,\ldots$). (2.3)

Then:

a) if series (2.2) converges, (2.1) converges as well;

b) if series (2.1) diverges, (2.2) diverges as well.

Remark. Theorem 2.1 is also valid in the case when inequalities (2.3) hold, starting with some number $n>N_0$.

In practice, the limit comparison test is more effective.

Theorem 2.2 (*limit comparison test*). Let $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ be positive terms series. If there exists a finite nonzero $\lim_{n\to\infty}\frac{u_n}{v_n}=k$ ($0<k<\infty$), then these series converge or diverge simultaneously.

Remark 1. The main disadvantage of using comparison test is the choosing of the reference series.

Note 2. Examining the convergence of the series $\sum_{n=1}^{\infty}\frac{P_m(n)}{Q_k(n)}$, where $P_m(n)$, $Q_k(n)$ are polynomials of degree m and k, accordingly, $m<k$, it is effective to apply the limit comparison test. The *Dirichlet-Riemann* series ($p=k-m>0$) should be taken as the reference series.

Theorem 2.3 (*D'Alembert's test*). Let $\sum_{n=1}^{\infty}u_n$ be the positive terms series and there exists $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l.$ Then given series converges for $l<1$ and diverges for $l>1$.

If $l=1$, then then the D'Alembert's test gives no answer as to whether a series converges or not. In this case, you need to use another test (for example, a comparison).

Remark. The D'Alembert's test should be used primarily for investigation of the convergence of positive terms series, the terms of which contain factorial or exponential functions.

Theorem 2.4 (*Cauchy's test*). Let $\sum_{n=1}^{\infty}u_n$ be the positive terms series and there exists finite or infinite limit $\lim_{n\to\infty}\sqrt[n]{u_n}=l.$ Then given series converges for $l<1$ and diverges for $l>1$.

If $l=1$ then the question of the convergence of the series remains open, i.e., requires additional research.

Remark 1. It is rationally to use Cauchy's test if you have to examine the convergence of positive terms series $\sum_{n=1}^{\infty}u_n$, which general term can be represented in the form $u_n=(f(n))^n$.

Remark 2. Investigating the series on convergence by the Cauchy's test, the following limits may be useful: $\lim_{n\to\infty}\sqrt[n]{a}=1$ ($a>0$), $\lim_{n\to\infty}\sqrt[n]{n}=1$.

Theorem 2.5 (*integral test*). Let the terms of the positive terms series $\sum_{n=1}^{\infty}u_n$ be the values of some continuous function $f(x)$ which is monotonically decreasing on the interval $[1;\infty)$ for natural values of the argument $x,$, i.e. $u_1 = f(1)$, $u_2 = f(2), \dots,$ $u_n = f(n), \dots$ Then the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)dx$ are simultaneously convergent or divergent.

Examples of solution of typical problems

Example 1. Use the comparison test to examine the following series for convergence:

a) $\sum_{n=1}^{\infty} \frac{3}{n+4}$; b) $\sum_{n=1}^{\infty} \frac{n}{n^5+3}$; c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{2n+5}}$; d) $\sum_{n=1}^{\infty} \arcsin\frac{3}{2n}$; e) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

Solution:

a) we use the limit comparison test taking divergent harmonic series for comparison $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$: $k = \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{3}{n+4} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{3n}{n+4} = 3$. Since $k = 3 \in (0;\infty)$ and harmonic series is divergent then given series is divergent;

b) the general term $u_n = \frac{n}{n^5+3}$ of given positive terms series is ratio of polynomials of the first and the fifth degree. The degree of denominator is 4 greater than the degree of the numerator. Therefore, for comparison, we choose a generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ with general term $v_n = \frac{1}{n^4}$. Let's apply limit comparison test:

$k = \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n}{n^5+3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^5}{n^5+3} = 1$. Since the calculated limit $0 < k < \infty$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent, then according to a comparison test the given series is convergent as well;

c) we have the positive terms series with general term $u_n = \dfrac{1}{\sqrt[4]{2n+5}}$.

Let's use a limit comparison test taking for comparison of divergent series $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\frac{1}{4}}}$ ($p = \dfrac{1}{4} < 1$):

$k = \lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[4]{2n+5}} \cdot \dfrac{\sqrt[4]{n}}{1} = \dfrac{1}{\sqrt[4]{2}}$. Since the calculated limit is positive number and chosen for comparison generalized harmonic series is divergent then according to a limit comparison test the given series is divergent as well;

d) we compare the given series with a divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ by the limit comparison test:

$$k = \lim_{n\to\infty} \frac{\arcsin\frac{3}{2n}}{\frac{1}{n}} = \left| \begin{array}{c} n \to \infty, \alpha = \dfrac{3}{2n} \to 0 \\ \arcsin\alpha \sim \alpha \end{array} \right| = \lim_{n\to\infty} \frac{\frac{3}{2n}}{\frac{1}{n}} = \frac{3}{2}.$$

Since $0 < k < \infty$ and the series for comparison is divergent than according to the limit comparison test given series is divergent as well;

e) let's use the comparison test (theorem 2.1), taking for comparison divergent harmonic series $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. For $n = 2; 3; \ldots$: $\ln n < n$ and correspondently $u_n = \dfrac{1}{\ln n} > \dfrac{1}{n} = v_n$. Since the terms of the given series greater than the respective terms of the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, then the given series is divergent.

Example 2. Examine the series for convergence using D'Alembert's test either Cauchy's test or an integral test:

a) $\sum\limits_{n=1}^{\infty} n \cdot \mathrm{tg}\dfrac{\pi}{2^n}$; b) $\sum\limits_{n=1}^{\infty} \dfrac{(n+1)^3 \cdot 3^n}{(2n+1)!}$; c) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n \cdot \left(\dfrac{n+5}{n}\right)^{n^2}$.

Solution:

a) we have the positive terms series which general term contains exponent function. Therefore, we use D'Alembert's test: $u_n = n \cdot \mathrm{tg}\dfrac{\pi}{2^n}$,

$u_{n+1} = (n+1) \cdot \mathrm{tg}\dfrac{\pi}{2^{n+1}}$,

$$l = \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{\mathrm{tg}\dfrac{\pi}{2^{n+1}}}{\mathrm{tg}\dfrac{\pi}{2^n}}\right) = \left|\begin{matrix}\alpha\to 0\\ \mathrm{tg}\,\alpha \sim \alpha\\ \dfrac{\pi}{2^n}\to 0\end{matrix}\right| = \lim_{n\to\infty}\left(1\cdot\frac{\dfrac{\pi}{2\cdot 2^n}}{\dfrac{\pi}{2^n}}\right) = \frac{1}{2} < 1$$

and correspondently the series is convergent;

b) if the general term of the series contains a factorial, it is recommended to use the D'Alembert's test: $u_n = \dfrac{(n+1)^3\cdot 3^n}{(2n+1)!}$,

$$u_{n+1} = \frac{(n+2)^3\cdot 3^{n+1}}{(2(n+1)+1)!};\ l = \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{(n+2)^3\cdot 3^{n+1}}{(2n+3)!}\cdot\frac{(2n+1)!}{(n+1)^3\cdot 3^n}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\frac{n+2}{n+1}\right)^3\cdot\frac{3\cdot(2n+1)!}{((2n+1)!)\cdot(2n+2)(2n+3)}\right) = \lim_{n\to\infty}\frac{3}{(2n+2)(2n+3)} = 0 < 1.$$

Hence, the series is convergent;

c) the general term of the given positive terms' series can be represented in the form $u_n = (f(n))^n$ then we use Cauchy's test:

$$l = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{2}\right)^n\left(\frac{n+2}{n}\right)^{n^2}} = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+5}{n}\right)^n = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = \frac{e^5}{2} > 1,$$

hence, the series is divergent;

Self-test questions

1. What series are called positive terms series?
2. How to investigate the positive terms series for convergence?
3. Give examples of reference series. How are reference series used in the investigation of positive terms series for convergence?

4. How is it recommended to investigate a series $\sum_{n=1}^{\infty}\frac{P_m(n)}{Q_k(n)}$ for convergence if $P_m(n)$, $Q_k(n)$ are polynomials of m and k correspondently, $m<k$?

5. Formulate the D'Alemfert's test. Which positive terms series is it applied to?

6. Formulate the Cauchy's test. Which positive terms series is it applied to?

7. Formulate the Integral test.

Self-test assignments

Task 1. Investigate the series on convergence using comparison test:

a) $\sum_{n=1}^{\infty}\frac{2n+1}{3n^2-1}$; b) $\sum_{n=1}^{\infty}\frac{n^2+n+1}{4n^5+3}$; c) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$;

d) $\sum_{n=1}^{\infty}\operatorname{tg}\frac{\pi}{3n+2}$; e) $\sum_{n=1}^{\infty}\frac{1}{\ln(n+4)}$; f) $\sum_{n=1}^{\infty}\frac{2^n}{3^n+2}$.

Task 2. Investigate the series on convergence using D'Alembert's test:

a) $\sum_{n=1}^{\infty}\sqrt{n}\operatorname{arctg}\frac{1}{2^n}$; b) $\sum_{n=1}^{\infty}\frac{4^n}{(n+2)!}$; c) $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\ldots(2n-1)}{2\cdot 5\cdot 8\cdot\ldots(3n-1)}$;

d) $\sum_{n=1}^{\infty}\frac{(2n+1)!}{n^3\cdot 7^{n+1}}$; e) $\sum_{n=1}^{\infty}2^n\sin\frac{4n+1}{3^n}$; f) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$.

Task 3. Investigate the series on convergence using Cauchy's test:

a) $\sum_{n=1}^{\infty}\left(\frac{2n+1}{5n+3}\right)^n$; b) $\sum_{n=1}^{\infty}\left(\frac{n+1}{3n+2}\right)^{2n+3}$; c) $\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n\left(\frac{n+2}{n}\right)^{n^2}$;

d) $\sum_{n=1}^{\infty}\arccos^n\left(\frac{n}{2n+1}\right)$; e) $\sum_{n=1}^{\infty}\frac{(n+1)^n}{2^n}$; f) $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n^2+1}\right)^n$.

Task 4. Investigate the series on convergence using Integral test:

a) $\sum_{n=1}^{\infty}\frac{1}{n(1+\ln^2 n)}$; b) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$; c) $\sum_{n=1}^{\infty}\frac{1}{(2n+1)\ln(2n+1)}$.

Answers: 1. a) divergent; b) convergent; c) divergent; d) divergent; e) divergent; f) convergent. **2.** a) convergent; b) convergent; c) convergent;

d) divergent; e) convergent; f) convergent. **3.** a) convergent; b) convergent; c) convergent; d) divergent; e) divergent; f) convergent **4.** a) convergent; b) convergent; c) divergent.

Topic 3. ALTERNATING SERIES

Plan

1. Types of number series.
2. Alternating series. Sufficient condition of convergence of alternating series.
3. Alternating Series. Leibniz' Test.
4. Absolute and Conditional Convergence.
5. Properties of the absolute convergent series.
6. Investigation of alternating series for absolute and conditional convergence.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 3 the student should ***know***: classification of numerical series, definition of alternating series, absolute and conditional convergence of alternating series, sufficient conditions of convergence of alternating series (Leibniz' test), basic properties of absolutely convergent series; ***be able to***: investigate the convergence of alternating series by the Leibniz test, to investigate alternating series for absolute and conditional convergence.

The basic theoretical information

Number series is called *alternating* series if it contains infinite number both positive and negative terms. The alternation of the sign can be both regular and chaotic. Examples of alternating series: $\sum_{n=1}^{\infty}\frac{\sin n}{n\cdot(n+2)};\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}.$

According to property 3 of a number series, the following statement is true: the convergence of a series does not depend on the rejection or addition of a finite number of terms.

Therefore, the investigation of number series containing both a finite number of positive (negative) and an infinite number of negative (positive) terms simultaneously is reduced to investigation respective positive term number series. Remark: the study of the convergence of numerical series, all members of which are negative, is reduced to the examining of the corresponding positive terms series, which is formed after taking out the “minus” sign of all terms in brackets.

Thus, the investigation of number series for convergence is reduced to the investigation of either positive terms or alternating series.

Theorem 3.1 (*sufficient condition for convergence of alternating series*). Let the number series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + ... + u_n + ...$ be alternating. If constructed from the modules of the terms of the given series the positive term series $\sum_{n=1}^{\infty} |u_n| = |u_1| + |u_2| + ... + |u_n| + ...$ converges then given alternating series converges as well.

Remark. Sufficient condition of convergence of alternating series is not necessary. That is, an alternating series $\sum_{n=1}^{\infty} u_n$ can be convergent even when the series with the absolute values of its terms $\sum_{n=1}^{\infty} |u_n|$ is divergent.

Let $\sum_{n=1}^{\infty} u_n$ be the alternating number series. Then the following statement are true:

1) $\sum_{n=1}^{\infty} |u_n|$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ is convergent;

2) $\sum_{n=1}^{\infty} |u_n|$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ is convergent or divergent;

3) $\sum_{n=1}^{\infty} u_n$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} |u_n|$ is convergent or divergent;

4) $\sum_{n=1}^{\infty} u_n$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} |u_n|$ is divergent.

Series, the signs of whose terms are strictly alternating:

$$u_1 - u_2 + u_3 - ... + (-1)^{n+1} u_n + ... = \sum_{n=1}^{\infty} (-1)^{n+1} u_n, \quad (3.1)$$

where $u_n > 0$ for $n \in N$.

Let's formulate sufficient condition of convergence for the series (3.1).

Theorem 3.2 (*Leibniz' Test*). Series (3.1) is convergent if:

1) $u_1 > u_2 > u_3 > ... > u_n > ...$;

2) general term of the series approaches to zero: $\lim_{n \to \infty} u_n = 0$.

Remark 1. First condition of the Leibniz' test can be fulfilled not from the first, but from some other term.

Remark 2. The series $\sum_{n=1}^{\infty} (-1)^n u_n = -u_1 + u_2 - ... + (-1)^n u_n + ...,$ where $u_n > 0$ for $n \in N$, is alternating too.

Remark 3. If alternating series (3.1) is convergent then the sum of the series S satisfies the condition $0 < S < u_1$.

Remark 4. From the Leibniz test it follows that for the convergent alternating series (3.1) the condition $|S - S_n| \le u_{n+1}$ is satisfied or $|r_n| \le u_{n+1}$, where $r_n = (-1)^n \cdot (u_{n+1} - u_{n+2} + u_{n+3} - ...)$. This property is used for approximate calculation of the sum of the alternating series with a given accuracy.

The alternating series $\sum_{n=1}^{\infty} u_n$ is called *absolutely convergent* if the series constructed from modules of its terms $\sum_{n=1}^{\infty} |u_n|$ is convergent.

The alternating series $\sum_{n=1}^{\infty} u_n$ is called *conditionally convergent* if the series constructed from modules of its terms $\sum_{n=1}^{\infty} |u_n|$ is divergent and given alternating series $\sum_{n=1}^{\infty} u_n$ is convergent.

Properties of absolute convergent series

1. If the alternating series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent, then its terms can be grouped and rearranged in any way. The series remains the same and its sum will not be changed.

2. Absolutely convergent series with sums S_1 and S_2 can be added and subtracted. The resulted series will also be absolutely convergent and its sum is equal to $S_1 \pm S_2$ accordingly.

3. The product of two absolutely convergent series with sums S_1 and S_2 is an absolutely convergent series which sum is equal to $S_1 \cdot S_2$.

Investigation for absolute and conditional convergence of the alternating series

All alternating number series can be classified according to the scheme represented in Figure 3.1.

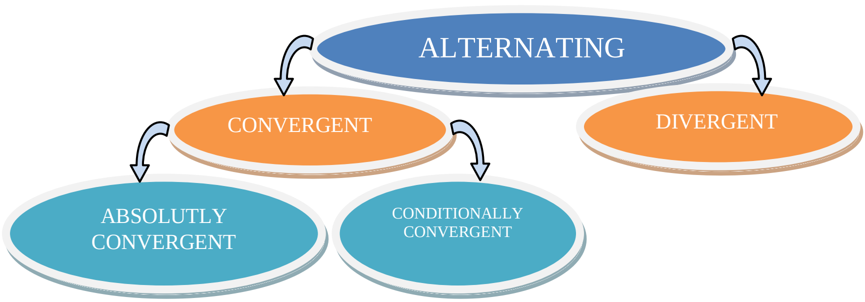

Figure. 3.1

To prove the absolute convergence of the alternating series $\sum_{n=1}^{\infty} u_n$, it is sufficient to represent the convergence of the positive terms series $\sum_{n=1}^{\infty} |u_n|$, formed from the modules of its terms.

To prove the conditional convergence of the alternating series $\sum_{n=1}^{\infty} u_n$, it is sufficient:

1) to represent the divergence of positive terms series $\sum_{n=1}^{\infty} |u_n|$;

2) to substantiate the convergence of the given alternating series $\sum_{n=1}^{\infty} u_n$, which is not always possible. If the alternating series has the form (3.1), then its convergence should be checked by the Leibniz' test.

Note. In practice, we advise you to start investigation for the conditional and absolute convergence of any alternating series by considering the corresponding positive terms series constructed from the modules of its terms.

Examples of solution of typical problems

Example 1. Examine for the absolute and conditional convergence the alternating series $\sum_{n=1}^{\infty} \frac{\sin n}{n \cdot (n+2)}$.

Solution.

Let's consider series constructed from modules of terms of given series: $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n \cdot (n+2)}$. Obtained positive terms series can be investigated for convergence by the comparison test. For comparison convergent series $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (Dirichlet-Riemann series for $p = 2 > 1$) can be taken. Since $0 \le |u_n| = \frac{|\sin n|}{n \cdot (n+2)} \le \frac{1}{n^2} = v_n$ for $n \in N$, then from convergence of the series $\sum_{n=1}^{\infty} v_n$ follows convergence of the series $\sum_{n=1}^{\infty} |u_n|$. Since the series $\sum_{n=1}^{\infty} |u_n|$, constructed from modules of its terms, is convergent then the alternating series $\sum_{n=1}^{\infty}\frac{\sin n}{n\cdot(n+2)}$ is absolutely convergent.

Example 2. Examine for the absolute and conditional convergence the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$.

Solution.

We have the alternative series $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot u_n,$ where $u_n=\frac{1}{n}>0$ for $n\in N,$ which is called Leibnitz' series.

Let's consider the series constructed from modules of the terms of the given $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}.$ The last series is harmonic which is divergent.

Hence, the given series can be only conditionally convergent. We investigate this alternating series by the Leibnitz' test:

1) $u_1>u_2>u_3>\ldots>u_n>\ldots,$ since $1>\frac{1}{2}>\frac{1}{3}>\ldots>\frac{1}{n}>\ldots$ for $n\in N$.

2) $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n}=0.$ Then the given series is convergent according to the Leibnitz' test.

Hence, the Leibnitz' series is conditionally convergent because it is convergent by the Leibniz' test and the series formed from the modules of its terms is divergent.

Example 3. Examine for the absolute and conditional convergence the alternating series $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{5^n}$.

Solution.

Let's consider the series constructed from the modules of terms of the given series: $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n}{5^n}\right|=\sum_{n=1}^{\infty}\frac{n}{5^n}.$ This series should be investigated by the D'Alembert's:

$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}=\frac{1}{5}\lim\limits_{n\to\infty}\frac{n+1}{n}=\frac{1}{5}<1.$ Therefore, the series constructed from terms modules converges. And initial series converges absolutely according to the definition of absolute convergence.

Example 4. Calculate the sum of the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{10^n n!}$ approximately with accuracy $\varepsilon=0,001.$

Solution.

We have convergent series (make sure of it yourself) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{10^n n!}=-\frac{1}{10}+\frac{1}{10^2\cdot 2!}-\frac{1}{10^3\cdot 3!}+\ldots+\frac{(-1)^n}{10^n\cdot n!}+\ldots,$ whose terms are strictly alternated. According to the corollary of the Leibniz's theorem the absolute error of replacing the sum of a convergent series (3.1) by its partial sum does not exceed the modulus of the first of the rejected terms of the series, i.e. $|r_n|=|S-S_n|\le u_{n+1}$.

Let’s find the smallest n starting from which the inequality holds

$u_{n+1}<\varepsilon,$ then $|r_n|<\varepsilon:$ $\frac{1}{10^2\cdot 2!}=\frac{1}{200}>\varepsilon,$ $\frac{1}{10^3\cdot 6}=\frac{1}{6000}<\varepsilon.$

Hence, $|r_2|<u_3<\varepsilon$, therefore to achieve this accuracy it is enough to take the sum of the first two terms of the series for: $S\approx-\frac{1}{10}+\frac{1}{200}=-0,1+0,005=-0,095.$

Self-test questions

1. Name the basic types of number series.
2. What series are called alternating?
3. How to investigate number series with arbitrary terms for convergence?
4. What series are called alternating? Formulate the Leibniz test.
5. Formulate the definition of absolute and conditional convergence of alternating series.
6. Formulate the basic properties of absolutely convergent series.

7. According to which algorithm alternating series are investigated for absolute and conditional convergence?

8. Justify whether the statements are correct for the alternating series $\sum_{n=1}^{\infty} u_n$: a) from convergence $\sum_{n=1}^{\infty} |u_n| \Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent; b) from divergence $\sum_{n=1}^{\infty} |u_n| \Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent; c) from divergence $\sum_{n=1}^{\infty} u_n \Rightarrow \sum_{n=1}^{\infty} |u_n|$ is divergent.

Self-test assignments

Task 1. Examine the series for absolute and conditional convergence:

a) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n+1}$; b) $\sum_{n=1}^{\infty} \frac{\sin 2^n}{2^n}$.

Task 2. Examine the alternating series for absolute and conditional convergence:

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+9}{11n-3}$; b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+3}{3n+2}\right)^n$; c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+n+1}$;

d) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$; e) $\sum_{n=1}^{\infty} 2^n \operatorname{tg} \frac{(-1)^n}{5^n}$; f) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$.

Task 3. Calculate the sum of series approximately with accuracy ε**,** noting the least sufficient number of terms of the series:

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \cdot n!}$, $\varepsilon = 0{,}001$; b) $\sum_{n=1}^{\infty} \left(-\frac{2}{13}\right)^{n+1}$, $\varepsilon = 0{,}001$.

Answers: 1. a) conditionally convergent; b) absolutely convergent; **2.** a) divergent; b) absolutely convergent; c) conditionally convergent; d) absolutely convergent; e) absolutely convergent; f) conditionally convergent. **3.** a) $S \approx 0{,}944$, $n = 3$; b) $S \approx 0{,}134$, $n = 3$.

Topic 4. FUNCTIONAL SERIES

Plan

1. Functional series. Basic concepts and definitions.
2. Uniform convergence. Weierstrass’ test.

3. Properties of uniformly convergent series.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 4 the student should ***know***: definition of functional series, absolute and uniform convergence of functional series, sufficient condition of uniform convergence of functional series (Weierstrass' test), properties of uniformly convergent series; ***be able to***: recognize the functional series, find the domain of absolute convergence of functional series, investigate the functional series for absolute and uniform the convergence by the Weierstrass' test.

The basic theoretical information

Expression of the form

$$u_1(x)+u_2(x)+...+u_n(x)+...=\sum_{n=1}^{\infty}u_n(x)\,, \qquad (4.1)$$

where $\left(u_n(x)\right)$ is sequence of functions, is called a functional series.

If you fix $x=x_0\in D$ in series (4.1) then the functional series becomes numerical. This series can converge or diverge. If a number series converges at a point x_0, then the point x_0 is called the point of convergence of the functional series.

The set of all values x for which the functional series is convergent is called the *domain of its convergence*.

The sum $S_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)$ is called *n-th partial sum* of the series (4.1). At each point x belonging to the domain of convergence, there exists the finite limit $\lim\limits_{n\to\infty}S_n(x)=S(x)$ which is called the sum of the series (4.1).

If the functional series (4.1) converges to function $S(x)$, then difference $r_n(x)=S(x)-S_n(x)$ is called *n-th* remainder of series: $r_n(x)=u_{n+1}(x)+u_{n+2}(x)+\ldots$ At the points of convergence the remainder of series tends to zero for $n\to\infty$: $\lim\limits_{n\to\infty}r_n(x)=0$.

The functional series (4.1) is called *absolutely convergent* if series $|u_1(x)|+|u_2(x)|+...+|u_n(x)|+...=\sum_{n=1}^{\infty}|u_n(x)|$ is convergent.

Sufficient tests of convergence of numerical series are used to find the domain of absolute convergence of a functional series. For example, according to the D'Alembert's test the limit $\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=l(x)$ should be found and then the inequality $l(x)<1$ should be solved.

Additionally, a series is investigated at points for which $l(x)=1$. Similarly, the functional series can be also investigated by the radical Cauchy's test.

Uniform convergence of the functional series

The functional series (4.1) is called a *uniformly convergent* series on the set D if for any number $\varepsilon>0$ there exists such number $N=N(\varepsilon)$ which depends on ε and doesn't depend on x, that for all $n>N$ and for all $x\in D$ the following inequality $|r_n(x)|<\varepsilon$ is true.

Sufficient Weierstrass' test is often used to examine the functional series for uniform convergence.

Theorem 4.1 *(Weierstrass' test*) A functional series (4.1) is absolutely and uniformly convergent on a set *D*, if there exists a convergent numerical series $\sum_{n=1}^{\infty}a_n$ with such positive terms that for all $x\in D$ the following inequalities are true $|u_n(x)|\le a_n$ ($n=1, 2, \dots$).

In this case, the series $\sum_{n=1}^{\infty}a_n$ is called *dominated* for the series (4.1), and the series (4.1) is called *correctly* convergent on the set D.

Properties of uniformly convergent series

1. If the functional series (4.1) is uniformly convergent on some interval *I* and the terms of this series are continuous functions on *I*, then the sum of this series is a continuous function on this interval.

2. If the functional series (4.1) is convergent on the interval *I*, its terms have continuous derivatives $u_n'(x)$ ($n=1, 2, \dots$) in this interval, and the series $\sum_{n=1}^{\infty} u_n'(x)$ is uniformly convergent on the interval *I*, then the given series can be differentiated term-by-term, i.e.

$$\left(\sum_{n=1}^{\infty} u_n(x)\right)' = \sum_{n=1}^{\infty} u_n'(x),\ x \in I.$$

3. If the functional series (4.1) is uniformly convergent on the interval I and the terms of the series are continuous functions on *I* then this series can be integrated term-by-term, i.e. on the interval $[\alpha;\beta] \in I$ the equality is fulfilled

$$\int_{\alpha}^{\beta} \left(\sum_{n=1}^{\infty} u_n(x)\right) dx = \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} u_n(x)dx.$$

The represented properties of uniformly convergent series can be used in approximate calculations.

Examples of solution of typical problems

Example 1. Find the domain of convergence of functional series $\sum_{n=1}^{\infty} \frac{1}{2^{nx} \cdot n}$.

Solution.

The given series is defined for any real x, and regardless of the x terms of this series are positive. Let's use the D'Alembert's test :

$$l(x) = \lim_{n\to\infty} \left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty} \frac{1}{2^{(n+1)x}(n+1)} \cdot \frac{2^{nx} n}{1} = \frac{1}{2^x} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{2^x}.$$

Since series converges for $l(x) < 1$ then the following inequality must be solved $\frac{1}{2^x} < 1,\quad 2^x > 1,\quad x > 0.$

The condition $l(x) = 1$ is fulfilled for $x = 0$ then we should investigate this series for convergence at this point: $\sum_{n=1}^{\infty} \frac{1}{2^{n\cdot 0} n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent series.

Hence, the domain of convergence of given series is $x \in (0;\, \infty)$.

Example 2. Find the domain of convergence of functional series $\sum_{n=1}^{\infty}\frac{1}{1+x^n}$.

Solution.

The series is defined for all values except the point $x=-1$.

Consider the cases:

1) $x=1$, then $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{2}=\frac{1}{2}\neq 0$, therefore the series at this point is divergent;

2) $-1<x<1$, then $\lim\limits_{n\to\infty}x^n=0$, $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{1+x^n}=1\neq 0$, therefore the series is divergent;

3) $x\in(-\infty;-1)\cup(1;\infty)$. At this case the series is convergent.

Really, since the series $\sum_{n=1}^{\infty}\frac{1}{x^n}$ is convergent for x satisfying the condition $\left|\frac{1}{x^n}\right|<1$, i.e. $|x|>1$ and $\lim\limits_{n\to\infty}\left|\frac{1}{1+x^n}\cdot x^n\right|=\lim\limits_{n\to\infty}\left|\frac{1}{x^{-n}+1}\right|=1$, then according to the limit comparison test this series is convergent for $|x|>1$.

Hence, domain of convergence of the original series is $x\in(-\infty;-1)\cup(1;\infty)$.

Self-test questions

1. What series are called functional?
2. Formulate the definition of the domain of convergence of the functional series.
3. Formulate the definition of absolute convergence of a functional series.
4. What functional series are called uniformly convergent?
5. How to find the domain of convergence of the functional series?
6. How to investigate the functional series for uniform convergence?
7. Formulate the basic properties of uniformly convergent series.

Self-test assignments

Task 1. Find the domain of convergence of functional series:

a) $\sum_{n=1}^{\infty}\frac{1}{2^{nx}}$; b) $\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$; c) $\sum_{n=1}^{\infty}\frac{1}{n^2+x^4}$; d) $\sum_{n=1}^{\infty}e^{(1-n)x}$; e) $\sum_{n=1}^{\infty}n\ln^n x$.

Task 2. Investigate the functional series for uniform convergence on the interval:

a) $\sum_{n=1}^{\infty}\frac{1}{5^n+x^2}$, $x\in(-\infty;\infty)$; b) $\sum_{n=1}^{\infty}e^{-nx}$, $x\in(0;\infty)$.

Answers: **1.** a) $(0;\infty)$; b) $x\neq\pm1$; c) $(-\infty;\infty)$; d) $(0;\infty)$; e) $(e^{-1};e)$.
2. a) converges uniformly; б) converges ununiformly.

Topic 5. POWER SERIES

Plan

1. Power series. The basic concepts and definitions.
2. Abel's Theorem. Interval and radius of convergence of power series.
3. Properties of power series.
4. Taylor's and Maclaurin's series.
5. An expansion of Elementary Functions into Maclaurin's Series.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 5 the student should ***know***: definition of power series, definition of radius and interval of convergence of power series, formulas of radius of convergence for complete power series, properties of power series, definition of Taylor's and Maclaurin's series; ***be able to***: recognize complete and incomplete power series, find the radius, interval and domain of convergence of power series, apply the properties of power series, decompose functions into the Maclaurin's series.

The basic theoretical information

The functional series of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots=\sum_{n=0}^{\infty}a_nx^n, \quad (5.1)$$

where $a_0, a_1, \dots, a_n, \dots$ are real numbers is called *power* series.

Functional series of the form

$$a_0 + a_1(x - x_0) + ... + a_n(x - x_0)^n + ... = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad (5.2)$$

where $a_0, a_1, \dots, a_n, \dots$ are real numbers, x_0 is some constant number, is a power series in powers of ($x - x_0$).

Domain of convergence of power series contains at least one point $x = 0$ for series (5.1) and $x = x_0$ for series (5.2).

Theorem 5.1 *(Abel'*s). If a power series (5.1) converges for some value $x = x_1 \neq 0$, then it converges absolutely for any value x satisfying the inequality $|x| < |x_1|$ (Fig 5.1,*a*)

Corollary. If a power series (5.1) diverges for some value $x = x_2$, then it diverges for any value x satisfying the inequality $|x| > |x_2|$. (Fig 5.1, b)

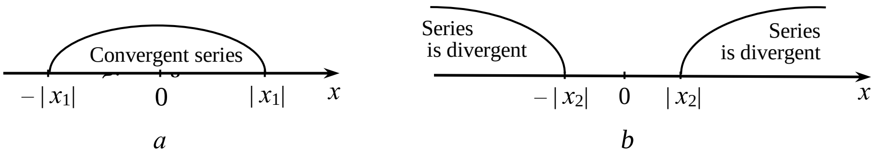

Fig. 5.1

There are three possible cases for series (5.1):

1) series is convergent only at one point $x = 0$;
2) series is convergent for any $x \in (-\infty; \infty)$;
3) there exists such positive number R that for $|x| < R$ series is absolutely convergent and for $|x| > R$ series is divergent (Fig. 5.2).

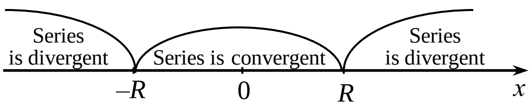

Fig.5.2

Number R is called *radius of convergence* of power series. Connection between radius and interval of convergence of power series (5.1) and (5.2) is represented in the table 5.1.

Radius of convergence of power series (5.1) and (5.2) are defined by the formulas:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|; \tag{5.3}$$

$$R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}. \tag{5.4}$$

Table 5.1

Radius of convergence R	Interval of convergence of power series (5.1)	Interval of convergence of power series (5.2)
$R=0$	$x=0$	$x=x_0$
$R=\infty$	$(-\infty;\ \infty)$	$(-\infty;\ \infty)$
$0<R<\infty$	$(-R;\ R)$	$(-R+x_0;\ R+x_0)$

Remark 1. If the power series does not contain all degrees x, i.e. it is incomplete, then the radius of convergence cannot be found directly by the formulas (5.3) and (5.4).

It is important to recognize incomplete power series and rationally choose the next steps investigating for their convergence. We should either define the interval of convergence by the D'Alembert's test or Cauchy's test as for a functional series or reduce the incomplete power series to complete series using the corresponding sabstitution (if it is possible).

Remark 2. If $0<R<\infty$ then in this case the power series can be convergent or divergent at the points that are the ends of the interval of convergence. Substituting points $x=-R; R$ to the series (5.1) or points $x=-R+x_0;\ R+x_0$ to the series (5.2) we investigate the formed number series for convergence.

Properties of power series

1. Power series (5.1) absolutely and uniformly converges on any segment $[-a;\ a]$ which lies completely within convergence interval $(-R;\ R)$.

2. The sum $S(x)$ of power series (5.1) is continuous function on the interval $(-R;\ R)$.

3. *(Differentiation of power series.)* A power series can be differentiated term-by-term within the interval of convergence. The series constructed by the differentiation has the same interval of convergence, moreover if $S(x)=\sum_{n=1}^{\infty}a_n x^n$, then $S'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$.

4. *(Integration of power series.)* The power series can be integrated term-by-term on any segment which lies within convergence interval $(-R;\ R)$. In particular, if the segment of integration $[0;x]\in(-R;\ R)$ and $S(x)=\sum_{n=1}^{\infty}a_n x^n$, then

$$\int_0^x S(x)dx=\int_0^x\left(\sum_{n=1}^{\infty}a_n x^n\right)dx=\sum_{n=1}^{\infty}\int_0^x a_n x^n dx=\sum_{n=1}^{\infty}a_n\frac{x^{n+1}}{n+1},$$

moreover the series constructed after integration has the same convergence interval.

5. The power series $\sum_{n=0}^{\infty}a_n x^n$ and $\sum_{n=0}^{\infty}b_n x^n$ with radii of convergence R_1 and R_2 respectively can be added, subtracted, multiplied. The radius of convergence of the formed series is not less than the smaller among numbers R_1 and R_2 .

Taylor's and Maclaurin's series

Suppose the function $f(x)$ is defined at some neighborhood of a point x_0 and has derivatives of all orders.

The series of the form

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+... \qquad (5.5)$$

is called *Taylor's series* of function $f(x)$.

Theorem 5.2. Taylor's series (5.5) converges to the function $f(x)$ in an interval $(x_0-R;\ x_0+R)$, i.e.

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+..., \qquad (5.6)$$

if and only if a function $f(x)$ has derivatives of all orders and expression $r_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$, where $c=x_0+\theta(x-x_0)$, $0<\theta<1$, tends to zero if $n\to\infty$ for all x from this interval: $\lim\limits_{n\to\infty} r_n(x)=0$, $x\in(x_0-R;x_0+R)$.

Particular case of Taylor's series for $x_0=0$ is called *Maclaurin's* series, which is expansion of function into power series in powers x:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots \quad (5.7)$$

Let's represent an expansion of some elementary functions into Maclaurin's series in table 5.2.

Table 5.2

№	Maclaurin's series of function $f(x)$	Domain of convergence
1	$e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^n}{n!}+\ldots$	$-\infty<x<\infty$
2	$\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}+\ldots$	$-\infty<x<\infty$
3	$\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots+(-1)^n\dfrac{x^{2n}}{(2n)!}+\ldots$	$-\infty<x<\infty$
4	$\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots+(-1)^{n-1}\dfrac{x^n}{n}+\ldots$	$-1<x\le 1$
5	$\dfrac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots$	$-1<x<1$
6	$\dfrac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+\ldots$	$-1<x<1$
7	$(1+x)^m=1+\dfrac{m}{1!}x+\dfrac{m(m-1)}{2!}x^2+$ $+\ldots+\dfrac{m(m-1)(m-2)\cdots(m-(n-1))}{n!}x^n+\ldots$	$-1\le x\le 1,\ m\ge 0;$ $-1<x\le 1,$ $m\in(-1;0);$ $-1<x<1,\ m\le -1.$

In practice, expansions of elementary functions represented in table 5.2 combined with the rules of addition, subtraction, multiplication of series and theorems on the integration and differentiation of power series are used to expand the functions into the Taylor's (Maclaurin's) series.

Examples of solution of typical problems

Example 1. Find the domain of convergence of power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+2)^n}{n\cdot 3^n}.$$

Solution.

We have complete power series then radius of convergence R can be calculated according to the formula (5.3):

$$a_n=\frac{(-1)^{n+1}}{n\cdot 3^n},\ a_{n+1}=\frac{(-1)^{n+2}}{(n+1)\cdot 3^{n+1}};$$

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{n\cdot 3^n}\cdot\frac{(n+1)\cdot 3^{n+1}}{(-1)^{n+2}}\right|=3\lim_{n\to\infty}\left|\frac{n+1}{n}\right|=3.$$

The interval of convergence is following: $-3<x+2<3,\ -5<x<1$. Then the series is absolutely convergent at internal points of the interval $(-5;1)$.

Let's investigate the behavior of a series at the ends of the convergence interval:

for $x=-5$ the initial series transforms to divergent series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-5+2)^n}{n\cdot 3^n}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n\cdot 3^n}{n\cdot 3^n}=\sum_{n=1}^{\infty}(-1)^{2n+1}\frac{1}{n}=-\sum_{n=1}^{\infty}\frac{1}{n};$$

for $x=1$ the series acquires the form: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n\cdot 3^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, which is conditionally convergent as Leibniz' series.

Hence, the domain of convergence of given series is interval $(-5;1]$.

Example 2. Find the domain of convergence of power series

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n x^{2n}.$$

Solution.

We have incomplete power series because it contains only even powers of x.

Denoting $x^2 = t \ge 0$ we obtain complete power series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n t^n$. Its radius of convergence is defined by the formula (5.4): $R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}} = \lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}} = \lim\limits_{n\to\infty}\frac{2n+1}{n} = 2.$

Constructed after substitution series converges for $t \in (-2;\, 2)$. Taking into account the limitations $t \ge 0$ we get $t \in [0;\, 2)$, that is, at the point $t = 0$ this series is convergent. Let's examine it at the right end of the convergence interval.

If $t = 2$ then we obtain the number series $\sum_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^n$. Since $\lim\limits_{n\to\infty}\left(\frac{2n}{2n+1}\right)^n = \lim\limits_{n\to\infty}\left(1 - \frac{1}{2n+1}\right)^n = \frac{1}{\sqrt{e}} \ne 0,$ then series diverges for $t = 2$. Hence, the series constructed after substitution is convergent at the interval $t \in [0;\, 2)$.

Returning to the substitution $x^2 = t$ we determine the domain of convergence of the initial series: $x^2 \in [0;\, 2),\ |x| < \sqrt{2},\ -\sqrt{2} < x < \sqrt{2}.$

Example 4. Expand the function into Maclaurin's series $f(x) = x\ln(2+x)$.

Solution.

Let's transform the function in following way:

$$f(x) = x\ln(2+x) = x\ln 2\left(1 + \frac{x}{2}\right) = x\ln 2 + x\ln\left(1 + \frac{x}{2}\right)$$

Using the expansion 4 from the table 5.2 for the function $\ln\left(1 + \frac{x}{2}\right)$ we get: $f(x) = x\ln 2 + x\left(x - \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3\cdot 2^3} - \ldots + (-1)^{n-1}\frac{x^n}{n\cdot 2^n} + \ldots\right) =$

$$= x\ln 2 + x^2 - \frac{x^3}{2\cdot 2^2} + \frac{x^4}{3\cdot 2^3} - \ldots + (-1)^{n-1}\frac{x^{n+1}}{n\cdot 2^n} + \ldots\ .$$

The domain of convergence of obtained series coincides to the domain of convergence of the Maclaurin’s series for the function $\ln\left(1+\frac{x}{2}\right): -1<\frac{x}{2}\le 1$, i.e. $-2<x\le 2$.

Self-test questions

1. What series are called power series?
2. Formulate the definition of the radius and interval of convergence of the power series.
3. How to find the radius of convergence of power series?
4. Demonstrate the connection between the radius, interval, and convergence domain of a power series.
5. How to find the domain of convergence of power series?
6. Formulate the basic properties of power series.
7. Formulate the definition of the Taylor’s and Maclaurin’s series.
8. How to expand functions into Maclaurin’s series?
9. Is it correct to say that not every power series is functional? Justify the answer.

Self-test assignments

Task 1. Find the domain of convergence of power series:

a) $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^3}$; b) $\sum\limits_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 2^{n+1}}$; c) $\sum\limits_{n=0}^{\infty}\frac{n!\cdot(x+3)^{n+1}}{7^{n-1}}$; d) $\sum\limits_{n=1}^{\infty}\frac{x^n}{\sqrt[4]{n+1}}$;

e) $\sum\limits_{n=0}^{\infty}\frac{x^{3n}}{8^n+1}$; f) $\sum\limits_{n=1}^{\infty}(2n+3)\cdot 5^n(x-1)^n$.

Task 2. Expand the function into Maclaurin’s series:

a) $\frac{1}{2x-3}$; b) $\frac{x}{1-x^3}$; c) $2x\cos^2 x$; d) $\ln(3+6x)$.

Answers: 1. a) $[-1;1]$; b) $[1;5)$; c) $\{-3\}$; d) $[-1;1)$; e) $(-2;2)$; f) $(-0,2;\ 0,2)$. **2.** a) $-\sum\limits_{n=0}^{\infty}\frac{(2x)^n}{3^{n+1}}$, $|x|<\frac{3}{2}$; b) $\sum\limits_{n=0}^{\infty}x^{3n+1}$, $x\in(-1;1)$; c) $2x-\frac{2^2x^3}{2!}+\frac{2^4x^5}{4!}-\ldots+(-1)^n\frac{2^{2n}x^{2n+1}}{(2n)!}+\ldots$, $x\in R$; d) $\ln 3+\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{2^n x^n}{n}$, $x\in(-0,5;\ 0,5]$.

Topic 6. APPLICATION OF POWER SERIES

Plan

1. Approximate calculation of function values.
2. Approximate calculation of definite integrals.
3. Approximate solution of differential equations.

Literature: [1]; [2]; [3]; [4]; [5]; [6].

Methodical guidelines

After studying the material of topic 6 the student should ***know***: definition of Taylor's and Maclaurin's series; ***be able to***: use power series to approximate the values of functions, definite integrals, and approximate the solution of differential equations satisfying the initial conditions.

The basic theoretical information

Power series are used to approximate the values of functions, definite integrals, approximate solution of differential equations satisfying the initial conditions, and so on.

Examples of solution of typical problems

Example 1. Calculate $\sqrt{e}$ with accuracy $\varepsilon = 0{,}005$.

Solution.

Using expansion of the function e^x into Maclaurin's series we receive $\sqrt{e} = e^{\frac{1}{2}} = 1 + \frac{1}{2\cdot 1!} + \frac{1}{2^2\cdot 2!} + \frac{1}{2^3\cdot 3!} + \ldots + \frac{1}{2^n n!} + \ldots$.

Let's define such n at which an error of the approximate equality

$$\sqrt{e} \approx 1 + \frac{1}{2\cdot 1!} + \frac{1}{2^2\cdot 2!} + \frac{1}{2^3\cdot 3!} + \ldots + \frac{1}{2^n n!}$$

doesn't exceed the given accuracy. For this purpose, we'll estimate the remainder

$$R_n = \frac{1}{2^{n+1}(n+1)!} + \frac{1}{2^{n+2}(n+2)!} + \frac{1}{2^{n+3}(n+3)!} + \ldots =$$

$$= \frac{1}{2^{n+1}(n+1)!}\left(1 + \frac{1}{2(n+2)} + \frac{1}{2^2(n+2)(n+3)} + \ldots\right) <$$

$$< \frac{1}{2^{n+1}(n+1)!}\left(1+\frac{1}{2(n+2)}+\frac{1}{2^2(n+2)^2}+\frac{1}{2^3(n+2)^3}+\ldots\right)=$$

$$=\frac{1}{2^{n+1}(n+1)!}\cdot\frac{1}{1-\frac{1}{2(n+2)}}=\frac{n+2}{2^n(2n+3)(n+1)!}.$$

We establish by means selection that inequality $R_n < \frac{n+2}{2^n(2n+3)(n+1)!} < 0,005$ is fulfilled starting with $n=3$.

Hence, $\sqrt{e} \approx 1+\frac{1}{2\cdot 1!}+\frac{1}{2^2\cdot 2!}+\frac{1}{2^3\cdot 3!} \approx 1+0,5+0,125+0,0208 \approx 1,646.$

Example 2. Calculate $\int_0^{\frac{1}{2}} e^{-x^2}\,dx$ with accuracy $\varepsilon = 0,001$.

Solution.

Note that the indefinite integral $\int e^{-x^2}\,dx$ is not expressed through elementary functions. For calculation of integral we expand the integrand in power series and use the property of term-by-term integration of power series.

We receive

$$\int_0^{\frac{1}{2}} e^{-x^2}\,dx = \int_0^{\frac{1}{2}}\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots+(-1)^n\frac{x^{2n}}{n!}+\ldots\right)dx=$$

$$=\left.\left(x-\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)\cdot n!}+\ldots\right)\right|_0^{\frac{1}{2}}=$$

$$=\frac{1}{2}-\frac{1}{2^3\cdot 3}+\frac{1}{2^5\cdot 5\cdot 2!}-\frac{1}{2^7\cdot 7\cdot 3!}+\ldots+(-1)^n\frac{1}{2^{2n+1}(2n+1)\cdot n!}+\ldots$$

Since obtained series is alternating and satisfies conditions of Leibniz' theorem then according to the corollary of this theorem the following inequality is true: $|r_n| \le u_{n+1}$. Let's determine the least number n for which the inequality $u_{n+1} < 0,001$ is true, i.e. $\frac{1}{2^{2n+1}(2n+1)\cdot n!} < 0,001$. This inequality is fulfilled starting from $n=3$:

$\frac{1}{2^7 \cdot 7 \cdot 3!} = \frac{1}{5376} < 0,001$. Therefore, we take the first three terms of the series and obtain

$$\int_0^{\frac{1}{2}} e^{-x^2} dx \approx \frac{1}{2} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5 \cdot 2!} \approx 0,5 - 0,0417 + 0,0031 \approx 0,461.$$

Example 3. Find the approximate solution of the Cauchy's problem $y' = x^2 + y^3$, $y(0) = 1$ using the first four nonzero terms of expansion of this solution in power series.

Solution.

We'll find the solution in the form of Maclaurin's series

$$y = y(0) + \frac{y'(0)}{1!} x + \frac{y''(0)}{2!} x^2 + \frac{y'''(0)}{3!} x^3 ... + \frac{y^{(n)}(0)}{n!} x^n + ... \ .$$

From the condition of the task we can find the first two coefficients $y(0) = 1, \ \ y'(0) = 0^2 + 1^3 = 1.$

We differentiate the initial equation $y'' = 2x + 3y^2 y'$.

Then we substitute $x = 0, \ y(0) = 1$ and $y'(0) = 1,$ into this equation and get the factor $y''(0) = 0 + 3 = 3$ Now we pass to the equation $y''' = 2 + 3(2y(y')^2 + y^2 y'').$ Then $y'''(0) = 2 + 3(2 + 3) = 17.$ So, the approximate solution of Cauchy's problem is defined by the formula $y \approx 1 + x + \frac{3}{2} x^2 + \frac{17}{6} x^3.$

This formula is more precise if x tends to zero.

Self-test questions

1. Formulate the definition of the Taylor's and Maclaurin's series.
2. How to decompose functions to the Maclaurin's series?
3. How to use power series for approximate calculation of the values of functions?
4. How to use power series for approximate calculation of definite integrals?
5. How to use power series for approximate solution of differential equations that satisfy the initial conditions?

Self-test assignments

Task 1. Calculate the value of function with accuracy ε :

a) $\sqrt[3]{130}$, $\varepsilon = 0,0001$; b) $\dfrac{1}{\sqrt[3]{e}}$, $\varepsilon = 0,001$; c) $\cos 10°$, $\varepsilon = 0,0001$.

Task 2. Calculate the definite integrals with accuracy ε :

a) $\int_0^{\frac{1}{2}} \frac{\sin x}{x} dx$, $\varepsilon = 0,0001$; b) $\int_0^{1} \cos x^2 dx$, $\varepsilon = 0,0001$; c) $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$, $\varepsilon = 0,001$; d) $\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{1+x^4}}$, $\varepsilon = 0,001$.

Task 3. Find the approximate solution of the Cauchy's problem using the first four nonzero terms of expansion of this solution in power series:

a) $y' = xy + e^y$, $y(0) = 0$; b) $y'' = yy' - x^2$, $y(0) = 1$, $y'(0) = 1$.

Answers: 1. a) 5,0658; b) 0,716; c) 0,9948. **2.** a) 0,4931; b) 0,9045; c) 0,494; d) 0,333. **3.** a) $y \approx 1 + \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{11}{24}x^4$; b) $y \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$.

Topic 7. FOURIER SERIES

Plan

1. Trigonometric Fourier series. Fourier coefficients.
2. Sufficient conditions for representation of the function by Fourier series.
4. Fourier series for even and odd functions.
5. Fourier series for 2π and $2l$ - periodic functions.

Literature: [1]; [2]; [3]; [4]; [5].

Methodical guidelines

After studying the material of topic 7 the student should ***know***: definition of trigonometric Fourier series and Fourier coefficients of the function $f(x)$, sufficient conditions for representation of the function $f(x)$ by Fourier series, complex form of Fourier series and complex coefficients of Fourier series; ***be able to***: recognize Fourier series, calculate Fourier coefficients and decompose 2π and $2l$-periodic functions into Fourier series.

The basic theoretical information

Functional series of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \ldots + a_n \cos nx + b_n \sin nx + \ldots =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (7.1)$$

is called *trigonometric series*. Constant numbers $a_0, a_1, b_1, \ldots, a_n, b_n, \ldots$ are called coefficients of trigonometric series. Absolute term is written in the form $\frac{a_0}{2}$.

Suppose the periodic function $f(x)$ with period 2π may be represented as a trigonometric series convergent to the given function within interval $[-\pi; \pi]$, i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad (7.2)$$

Numbers a_0, a_n, b_n, defined by the formulas

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx,$$

are called *Fourier coefficients* of function $f(x)$.

Trigonometric series (7.2) which coefficients are Fourier coefficients of function $f(x)$ is called Fourier series of function $f(x)$.

For integrable function $f(x)$ on a segment $[-\pi; \pi]$ we write:

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. The sign (~) denotes that Fourier series corresponds the function $f(x)$ integrable on a segment $[-\pi; \pi]$.

The following theorem gives sufficient conditions for representation of a function $f(x)$ by Fourier series

Theorem 7.1 (*Dirichlet's*). Suppose the following conditions are fulfilled for 2π - periodic function $f(x)$ on segment $[-\pi; \pi]$:

1) $f(x)$ is a piecewise continuous (continuous or has finite number of finite discontinuities),

2) $f(x)$ is monotone on segment $[-\pi; \pi]$ or this segment can be divided into a finite number of intervals so that on each of them the function is monotonic.

Then Fourier series of a function $f(x)$ is convergent everywhere and its sum $S(x)$ has the following properties:

1) $S(x) = f(x),$ at points of continuity $f(x)$, that is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx);$$

2) if x_0 is a point of discontinuity of the first type of a function $f(x)$, then

$$S(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2},$$

3) $S(-\pi) = S(\pi) = \dfrac{f(-\pi + 0) + f(\pi - 0)}{2}$.

Remark 1. For any integrable 2π-periodic function the Fourier coefficients can be calculated by the formulas:

$$a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nxdx, \quad b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nxdx,$$

where a is arbitrary real number.

Remark 2. To calculate the Fourier coefficients the following equalities are used: $\sin n\pi = 0$, $\cos n\pi = \left(-1\right)^n$, $n = 0, 1, 2, \ldots$.

Fourier series for odd and even functions

If $f(x)$ is either odd or even function then the calculation of the Fourier coefficients is simplified. At the same time, the form of the Fourier series is also simplified, it becomes incomplete (Table 7.1).

Table 7.1

Property of a function $f(x)$	Fourier series	Fourier coefficients
$f(x)$ is even function	$\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos nx$	$b_n=0,\ \ a_0=\dfrac{2}{\pi}\int\limits_0^{\pi}f(x)dx$, $a_n=\dfrac{2}{\pi}\int\limits_0^{\pi}f(x)\cos nxdx$
$f(x)$ is odd function	$\sum\limits_{n=1}^{\infty}b_n\sin nx$	$a_0=0,\ a_n=0,$ $b_n=\dfrac{2}{\pi}\int\limits_0^{\pi}f(x)\sin nxdx$

Fourier series for 2l-periodic functions

Suppose the function $f(x)$ is defined on the segment $[-l; l]$, and has the period $2l$ $(l>0)$. Suppose this function obeys all conditions of Dirichlet's theorem. Therefore, the Fourier series for $f(x)$ looks like

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi nx}{l}+b_n\sin\frac{\pi nx}{l}\right), \tag{7.3}$$

Where Fourier coefficients are defined by the formulas:

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx,\ a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{\pi nx}{l}dx,\ b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{\pi nx}{l}dx. \tag{7.4}$$

Table 7.2

Property of a function $f(x)$	Fourier series	Fourier coefficients
$f(x)$ is even function	$\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos\dfrac{\pi nx}{l}$	$b_n=0,\ \ a_0=\dfrac{2}{l}\int\limits_0^{l}f(x)dx,$ $a_n=\dfrac{2}{l}\int\limits_0^{l}f(x)\cos\dfrac{\pi nx}{l}dx$
$f(x)$ is odd function	$\sum\limits_{n=1}^{\infty}b_n\sin\dfrac{\pi nx}{l}$	$a_0=0,\ a_n=0,$ $b_n=\dfrac{2}{l}\int\limits_0^{l}f(x)\sin\dfrac{\pi nx}{l}dx$

Fourier series and Fourier coefficients for even and odd functions defined on segment $[-l;\, l]$ are given in Table 7.2.

Examples of solution of typical problems

Example 1. Expand 2π-periodic function in Fourier series $f(x)=|x|,\;\; f(x+2\pi)=f(x)$ (Fig. 7.2).

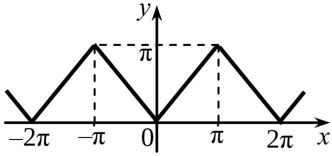

Fig. 7.2

Solution.

The given function satisfies to all conditions of Dirichlet's theorem. It is even function. Then Fourier series for this function looks like $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty} a_n\cos nx$.

Let's define Fourier coefficients a_0 and a_n (Table 7.1):

$$a_0=\frac{2}{\pi}\int_0^{\pi} x\,dx=\frac{2}{\pi}\frac{x^2}{2}\Bigg|_0^{\pi}=\pi; \qquad a_n=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx=\frac{2}{\pi}\left(x\frac{\sin nx}{n}\Bigg|_0^{\pi}\right)-$$

$$-\frac{2}{\pi}\left(\int_0^{\pi}\frac{\sin nx}{n}dx\right)=\frac{2}{\pi}\frac{\cos nx}{n^2}\Bigg|_0^{\pi}=\frac{2}{\pi n^2}((-1)^n-1).$$

That is Fourier series for the given function is

$$f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right).$$

The given function $f(x)$ is continuous everywhere. Therefore obtained representation is true for all $x\in R$.

Example 2. Expand function $f(x)=x^2$, $x\in[0;\pi]$ in Fourier series in sines.

Solution.

Let's extend $f(x)$ in odd way on $[-\pi;0)$, and then extend it periodically with period 2π on all numerical axis (Fig. 7.3). Function

$f(x)$ is odd on a segment $[-\pi; \pi]$. Therefore $a_0 = a_n = 0$. Let's find a coefficient b_n using formula $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$.

We get

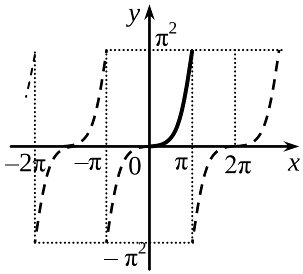

Fig. 7.3

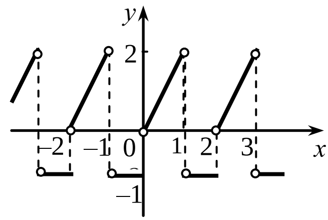

Fig. 7.4

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx dx = \frac{2}{\pi}\left(x^2 \cdot \frac{-\cos nx}{n}\Bigg|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx dx \right) =$$

$$= \frac{2}{\pi}\left(-\pi^2 \cdot \frac{\cos n\pi}{n} + \frac{2}{n}\left(x \cdot \frac{\sin nx}{n}\Bigg|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx dx \right)\right) =$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\cos nx\Bigg|_0^{\pi} \right) = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left((-1)^n - 1\right).$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{2\pi}{n}(-1)^{n+1}\sin nx + \sum_{n=1}^{\infty}\frac{4}{\pi n^3}\left((-1)^n - 1\right)\sin nx =$

$$= 2\pi\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots \right) - \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots \right).$$

This equality is fulfilled for $x \in [0; \pi]$, except point $x = \pi$, in which the sum of series equals 0, but $f(\pi) = \pi^2$.

Self-test questions

1. Formulate the definition of a trigonometric Fourier series.

2. Write down the formulas of the Fourier coefficients of the periodic function $f(x)$ with the period 2π.

3. Formulate sufficient conditions for the representation of a function $f(x)$ by Fourier series.

4. Write down the complex form of the Fourier series and the complex coefficients of the Fourier series.

5. Write down the formulas of Fourier coefficients and Fourier series for even and odd functions.

6. Write down the formulas of Fourier coefficients and Fourier series for 2π and $2l$-periodic functions.

Self-test assignments

Task 1. Expand 2π-periodic functions given in the interval $(-\pi;\pi)$ in the Fourier series.

a) $f(x)=x$; b) $f(x)=\begin{cases}3, \; if \;\; x\in(-\pi;0),\\ -1, \; if \;\; x\in[0;\pi);\end{cases}$ c) $f(x)=1+\dfrac{x}{2}$.

Task 2. Expand functions given in the interval $(0;\pi)$ in a Fourier series by cosines:

a) $f(x)=x^2$; b) $f(x)=\begin{cases}1, \; if \;\; x\in(0;\pi/2),\\ 0, \; if \;\; x\in[\pi/2;\pi).\end{cases}$

Task 3. Expand functions in Fourier series:

a) $f(x)=\begin{cases}1, \; if \;\; x\in(-1;0),\\ -1, \; if \;\; x\in[0;1);\end{cases}$ b) $f(x)=\begin{cases}0, \; if \;\; x\in(-3;1),\\ x, \; if \;\; x\in[1;3).\end{cases}$

Answers: 1. a) $2\sum\limits_{k=1}^{\infty}(-1)^{k+1}\dfrac{\sin kx}{k}$; b) $1-\dfrac{4}{\pi}\sum\limits_{k=1}^{\infty}\dfrac{\sin(2k-1)x}{2k-1}$; c) $1+\sin x-$ $-\dfrac{\sin 2x}{2}+\dfrac{\sin 3x}{3}-\dfrac{\sin 4x}{4}+\ldots$.**2.**a) $\dfrac{\pi^2}{3}+4\sum\limits_{k=1}^{\infty}(-1)^k\dfrac{\cos kx}{k^2}$;

b) $\dfrac{1}{2}+\dfrac{2}{\pi}\sum\limits_{k=1}^{\infty}(-1)^{k+1}\dfrac{\cos(2k-1)x}{2k-1}$. **3.**a) $-\dfrac{4}{\pi}\sum\limits_{k=1}^{\infty}\dfrac{\sin(2k-1)\pi x}{2k-1}$;

б) $\dfrac{2}{3}+\dfrac{1}{3}\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{\pi n}\sin\dfrac{\pi n}{3}+\dfrac{9}{\pi^2n^2}\left((-1)^n-\cos\dfrac{\pi n}{3}\right)\right)\cos\dfrac{\pi nx}{3}+$

$\dfrac{1}{3}\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{\pi n}\left(3(-1)^{n+1}+\cos\dfrac{\pi n}{3}\right)+\dfrac{9}{\pi^2n^2}\sin\dfrac{\pi n}{3}\right)\sin\dfrac{\pi nx}{3}$.

LIST OF REFERENCES

1. Дубовик В.П., Юрик І.І. Вища математика: Навч. посібник. – К.: А.С.К., 2001. – 681с.

2. Вища математика: Збірник задач: Навч. посібник / В.Дубовик, І. Юрик, І. Вовкодав та ін.; За ред. В.Дубовика, І. Юрика. – К: 2001 – 480 с.

3. Ластівка І.О., Безверхий О.І., Кудзіновська І.П. Вища математика: навч. Посібник. – К.: НАУ, 2018. – 452 с.

4. Denisiuk V. P. Mathematical analysis: Manual/ V.P. Denisiuk, V.G. Demydko., O.V. Karupu, T.A. Oleshko, V.V. Pakhnenko, V.K. Repeta.– Kyiv: NAU, 2016. – 396 p..

5. Denisiuk V. P., Grishina L. I., Karupu O. V. [and Others]. Higher mathematics. Part 3: Manual / V. P. Denisiuk, L. I. Grishina, O. V. Karupu, [and Others]. — K. : NAU, 2006. — 232 p.

Educational edition

HIGHER MATHEMATICS
NUMBER AND FUNCTIONAL SERIES

A guide
to independent work of higher education applicants
of technical specialties

Contributors:
LASTIVKA Ivan Oleksiyovich
LIASHENKO Yana Grygorivna

Technical editor
Proofreaders
Computer typesetting

Підп. до друку __.__.18. Формат 60х84/16. Папір офс.
Офс. друк. Ум. друк. арк. 2,25. Обл.-вид. арк. 2,65.
Тираж 100 прим. Замовлення № ____.

Видавець і виготовлювач
Національний авіаційний університет
03680. Київ-58, проспект Космонавта Комарова, 1.
Свідоцтво про внесення до Державного реєстру ДК № 977 від 05.07.2002