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ON APPLICATIONS OF MEASURE OF NONCOMPACTNESS IN FRÉCHET SPACES

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Abstract

In metric and topological vector spaces the notion of measure of noncompactness is used to associate numerical values to sets so that compact sets get zero measures and other ones obtain positive values that indicate how far they are different from compact sets. This concept was initiated by Kuratowski in early 30s and has been defined and developed in many different ways. The indices of noncompactness can give us sufficient conditions for formulating various fixe point theorems in metric spaces. Another important application of these measurements is in characterization of Fredholm operators in infinite dimensional topological vector spaces. The object of this paper is to provide an appropriate criterion that establishes a connection between Lipschitz-Fredholm operators in more general context of Fréchet spaces, the Hausdorff and lower measures of noncompactness. Furthermore, by using an arbitrary measure of noncompactness in the sense of Banas and Goebel we obtain a fixed point theorem for Fréchet spaces.

Keywords: Fréchet spaces; Fredholm operators; measure of noncompactness

1. Introduction

Fredholm operators are fundamental objects in integral and partial differential equations in Banach spaces and they have been the subject of study by many authors. There are several criteria for the Fredholmness one of which is various measure of noncompactness, cf. [1]. In attempting to generalize the Fredholm theory to more general Fréchet spaces we face some difficulties due to the lack of suitable topologies on the families of linear operators. In general, for Frechet spaces

– the space of linear isomorphism $E \to F$ may not be open in the space of linear continuous maps $CL(E,F)$;

– the space of linear Fredholm operators may not be open in $CL(E,F)$;

– there may exist continuous families of linear projections in which the dimensions of the kernel and cokernels vary quickly by finite or infinite amounts.

Thus, we need to impose some restrictions on operators. In this article we consider Lipschitz-Fredholm operators between Frechet spaces and give the criterion of Fredholmness via the Hausdorff and lower measures of noncompactness.

Another application of measure of noncompactness is to obtain fixed point theorems. Using the ideas from Banach spaces (cf. [2]) we generalize naturally a result to Fréchet spaces.

2. Fredholmness and measure of noncompactness

We recall that a Fréchet space (F,d) is a Hausdorff locally convex topological space whose topology is defined by a complete translational-invariant metric d.

We let $\|f\|_d := d(f,0)$ for $f \in F$ and write Lf instead of $L(f)$ when L is a linear map between Frechet spaces. Suppose (E,g) is another Fréchet space and $\Gamma_{g,d}(E,F)$ is the set of all globally linear Lipschitz continuous maps away from zero, i.e. linear maps $L: E \to F$ such that

$$\|L\|_{g,d} := \sup_{x \in E \setminus \{0\}} \frac{\|L\|_d}{\|x\|_g} < \infty.$$

We abbreviate $\Gamma_g(E) := \Gamma_{g,g}(E,E)$ and write $\|L\|_g = \|L\|_{g,g}$ for $L \in \Gamma_g(E)$.

Let F^* be the dual space of F endowed with the strong topology. Given an operator $L \in \Gamma_{g,d}(E,F)$ we denote by $L^* \in \Gamma_{g,d}(E^*,F^*)$ its strong dual defined by $(L^*\phi)x = \phi(Lx)$ for every $\phi \in F^*$, $x \in E$.

Definition 2.1. *[3]* A map $\varphi \in \Gamma_{g,d}(E,F)$ *is called Lipschitz-Fredholm operator if it satisfies the following properties:*

(1) *the image of* φ *is closed;*

(2) *the dimension of the kernel of* φ, $\dim\ker\varphi$, *is finite;*

(3) *the co-dimension of the image of* φ, $\dim\operatorname{coker}\varphi$, *is finite.*

We say that φ is upper (lower) semi Lipschitz-Fredholm operator if its image is closed and ker φ (coker φ) is of finite dimension. The index of φ is defined as follows:

$$\operatorname{Ind}\varphi = \dim\ker\varphi - \dim\operatorname{coker}\varphi.$$

Recall that if B is a bounded subset of F, then the Hausdorff measure of noncompactness of B is denoted by $\chi(B)$, and

$$\chi(B) = \inf\{\varepsilon > 0 : B \text{ has a finite } \varepsilon - \text{net in } F\}.$$

Let $L : E \to F$ be a globally Lipschitz operator. From the definition of the Hausdorff measure of noncompactness it follows that

$$\chi(L(M)) \le k\chi(M) \qquad (2.1)$$

for any bounded subset M of E, where at least $k = \|L\|_{g,d}$.

Let $m_1,\dots,m_n$ be a finite ε-net for M, then $Lm_1,\dots,Lm_n$ is a finite $\|L\|_{g,d}$ ε-net for $L(M)$. More generally, we define the Hausdorff measure of noncompactness of L by

$$[L]_A := \inf\{k : k > 0, \chi(L(M)) \le k\chi(M)\}. \qquad (2.2)$$

We also define the lower measure of noncompactness by

$$[L]_a := \inf\{k : k > 0, \chi(L(M)) \ge k\chi(M)\}.$$

When E is infinite dimensional we can write the Equations (3.1), (2.2) in the equivalent form

$$[L]_A := \sup_{\chi(M)>0} \frac{\chi(L(M))}{\chi(M)},$$

$$[L]_a := \inf_{\chi(M)>0} \frac{\chi(L(M))}{\chi(M)}. \qquad (2.3)$$

We should mention that in the case of finite dimensional spaces these equations do not make sense because all bounded sets are precompact and therefore there are no sets M satisfying $0 < \chi(M) < \infty$

A subset G of a Fréchet space E is called topologically complemented or split in E if there is another subspace H of E such that E is homeomorphic to the topological direct sum $G \oplus H$. In this case we call H a topological complement of G in F.

Theorem 2.1. *[4] Let* E *be a Fréchet space. Then*

(1) *Every finite-dimensional subspace of* E *is closed.*

(2) *Every closed subspace* $G \subset E$ *with* $\operatorname{codim}(G) = \dim(E/G) < \infty$ *is topologically complemented in* E.

(3) *Every finite-dimensional subspace of* E *is topologically complemented.*

(4) *Every linear isomorphism between the direct sum of two closed subspaces and* E, $G \oplus H \to E$, *is a homeomorphism.*

Theorem 2.2. *If* $L \in \Gamma_{g,d}(E,F)$ *and* $\lambda \in$ □, *then*

(1) $[L]_a > 0$ *if and only if* L *is upper semi Lipschitz-Fredholm operator.*

(2) $[L^*]_a > 0$ *if and only if* L *is lower semi Lipschitz-Fredholm operator.*

(3) L *is Lipschitz-Fredholm if and only if both* $[L]_a > 0$ *and* $[L^*]_a > 0$.

Proof. (1) Suppose that $[L]_a > 0$ and choose a fixed number $k \in (0,[L]_a)$. Let $B_1(E)$ be the closed unit ball with center 0 in E. The set $S := \ker(L) \cap B_1(E)$ is mapped into $\{0\}$, therefore, $\chi(S) \le \frac{1}{k}\chi(\{0\}) = 0$. Now we prove that the range $\operatorname{Img}(L)$ of L is closed. Since $\dim\ker L < \infty$ it follows that by Theorem 2.1 there exists a closed subset $E_0 \subseteq E$ such that

$E = E_0 \oplus \ker(L)$. Let (y_n) be a sequence in $\operatorname{Img}(L)$ converging to some y, and choose x_n in E with $Lx_n = y_n$. Now we distinguish two cases. First assume that x_n is bounded, then we obtain

$$\chi(\{x_n\}) \le \frac{1}{k}\chi(\{y_n\}) = 0$$

and hence $x_{n_k} \to x$ for some subsequence x_{n_k} of x_n and suitable $x \in E$. Continuity implies that $Lx = y$ and therefore $y \in \operatorname{Img}(L)$ so $\operatorname{Img}(L)$ is closed.

Now suppose that $\|x_n\|_g \to \infty$. Put $z_n := x_n / \|x_n\|_g$ and $G := \{z_1, z_2, \ldots\}$. Then $G \subset \partial B_1(E)$ and

$$Lz_n = \frac{Lx_n}{\|x_n\|_g} = \frac{y_n}{\|x_n\|_g} \to 0, \quad (n \to \infty)$$

thus $\chi(L(G)) = 0$. On the other hand by Definition 2.2 we have $\chi(L(G)) \ge k\chi(G)$ and hence $\chi(G) = 0$. Without loss of generality we may assume that the sequence z_n converges to some element $z \in \partial B_1(E_0)$. Therefore $Lz = 0$, contracting the fact that $\chi(S) \le \frac{1}{k}\chi(\{0\}) = 0$. Thus, L is upper semi Lipschitz-Fredholm operator.

Now we prove that the closedness of $\operatorname{Img}(L)$ and the fact that $\ker(L)$ is finite dimensional imply that $[L]_a > 0$. Since $\dim\ker(L) < \infty$ it follows that by Theorem 2.1 that there exists a closed subset $E_0 \subseteq E$ such that $E = E_0 \oplus \ker(L)$. The projection $Pr: E \to E_0$ satisfies $[Pr]_a = 1$. Consider the canonical isomorphism $\bar{L}: E_0 \to \operatorname{Img}(L)$. Since $L := \bar{L}\,Pr$ and $[\bar{L}]_a > 0$, we conclude that $[L]_a \ge [\bar{L}Pr]_a$ and directly by Definition 2.2 we obtain $[L]_a \ge [\bar{L}]_a [Pr]_a > 0$.

(2) Suppose that $[L^*]_a > 0$, by the above arguments we obtain $\operatorname{Img}(L^*)$ is closed and hence $\operatorname{Img}(L)$ is closed. Furthermore, the kernel $\ker(L^*)$ of L^* is finite dimensional hence we may find a basis $\{g_1, \ldots, g_n\}$ for $\ker(L^*)$. But the fact

$$\operatorname{Img}(L) = \ker(g_1) \cap \ldots \cap \ker(g_n)$$

shows that $\ker(L)$ has finite codimension. Thus, L is lower semi Lipschitz-Fredholm operator.

Now we show that the closedness of $\operatorname{Img}(L)$ and the fact that $\operatorname{Img}(L)$ has finite codimension imply that $[L^*]_a > 0$. Again the closedness of $\operatorname{Img}(L)$ implies the closedness of $\operatorname{Img}(L^*)$. On the other hand, we have $\operatorname{Img}(L) = \ker(g_1) \cap \ldots \cap \ker(g_n)$, where $\{g_1, \ldots, g_n\}$ is a basis of $\ker(L^*)$. Thus $\dim\ker(L^*) < \infty$, and the result proved in the second part of (1) implies that $[L^*]_a > 0$.

(3) It follows from (1) and (2).

3. A fixed point theorem and measure of noncompactness

We adapt the definition of measure of noncompactness by Banas and Goebel [5] for a Fréchet space *F*. Let B_F be the set of all nonempty and bounded subsets of F and Σ_F the set of all relatively compact subsets of F.

Definition 3.1. *[5] A convex function m:* $\mathrm{B}_F \to \mathbb{R}_+$ *is called measure of noncompactness if it satisfies*

(1) $\varnothing \ne \ker m \subset \Sigma_F$;

(2) $A \subset B \Rightarrow m(A) \subset m(B)$;

(3) $m(\bar{A}) = m(\operatorname{conv}(A)) = m(A)$;

(4) *if* A_n *is a sequence of closed sets in* B_F *such that* $A_{n+1} \subset A_n$ *and* $\lim_{n\to\infty} m(A_n) = 0$ *then* $A_\infty = \bigcap_{n=1}^{\infty} A_n \ne \varnothing$.

Theorem 3.1 *[6]*(Tychonoff's theorem)**.** Let A be a compact convex subset of a locally convex (linear topological) space and f a continuous map of A into itself. Then f has a fixed point.

Let f, g be respectively continuous and semi-continuous maps from $[0,\infty)$ to itself such that they are zero only at zero.

Theorem 3.2. *Let* $U \subset F$ *be bounded, closed and convex and let* $\varphi: U \to U$ *be a continuous function. If for each nonempty set* $A \subset U$

$$f(m(\varphi(A))) \le f(m(A)) - g(m(A)).$$

Then p at least has one fixed point.

Proof. Define a sequence $A_n \subset \mathrm{B}_F$

$$A_0 = A, \quad A_{n+1} = \overline{\varphi(A_n)}, \forall n \in \mathbb{N}.$$

Note that this sequence is ordered by inclusion. By Definition (3.1) we can find positive real number a such that

$$\lim_{n\to\infty} m\left(A_{n+1}\right)=a \qquad (3.1)$$

By our assumption and Definition (3.1) we obtain

$$f\left(m\left(A_{n+1}\right)\right)=f\left(m\left(\overline{conv\left(\varphi\left(A_n\right)\right)}\right)\right)=$$
$$=f\left(m\left(\varphi\left(A_n\right)\right)\right)\leq f\left(m\left(A_n\right)\right)-g\left(m\left(A_n\right)\right).$$

So $f\left(m\left(A_{n+1}\right)\right)\subset f\left(m\left(A_n\right)\right)-g\left(m\left(A_n\right)\right)$. Letting $n\to\infty$ yields

$$\limsup_{n\to\infty}\left(g\left(m\left(A_{n+1}\right)\right)\right)\leq$$
$$\leq\limsup_{n\to\infty}\left(g\left(m\left(A_n\right)\right)\right)-\liminf_{n\to\infty}\left(g\left(m\left(A_{n+1}\right)\right)\right)$$

Then using (3.1) we get $f\left(a\right)=f\left(a\right)-g\left(a\right)$ so $f\left(a\right)=0$ and hence $a=0$. That is

$$\lim_{n\to\infty} m\left(A_{n+1}\right)=0$$

By Definition 3.1 $A_\infty=\bigcap_{n=0}^{n=\infty}A_n$ is not empty closed and convex and $A_\infty\subset A$. Now by Tychonoff's theorem $\varphi: A_\infty\to A_\infty$ has a fixed point.

4. Conclusions

It was shown in Theorem 2.2 that the measures of noncompactness can be applied to investigate the Fredholmness of operators between metrizable locally convex spaces. It is not easy in practice to check by definition whether an operator is Fredholm so such results are very useful. In addition, as it was obtained in Theorem 3.2, beyond Banach spaces we can use the method of measure of noncompactness to formulate various fixed point theorems in more general context of locally convex spaces.

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Про застосування міри некомпактності в просторах Фреше

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У метричних і топологічних векторних просторах поняття міри некомпактності використовується для відповідності числових значень множинам так, що компактні множини отримують нульові міри, а інші - позитивні значення, які показують, наскільки вони відрізняються від компактних. Ця концепція була ініційована Куратовський на початку 30-х років, і була визначена та розроблена багатьма різними способами. Міри некомпактності можуть дати нам достатні умови для формулювання різних теорем про нерухомі точки в метричних просторах. Інша важливе застосування цих мір полягає в характеризації операторів Фредгольма в нескінченновимірних топологічних векторних просторах. Метою даної роботи є створення відповідного критерію, який встановлює зв'язок між операторами Ліпшиця-Фредгольма в більш загальному контексті просторів Фреше і міри некомпактності Хаусдорфа. Крім того, використовуючи довільну міру некомпактності в сенсі Банаса і Гебеля, ми отримуємо теорему про нерухому точку для просторів Фреше.

Ключові слова: простори Фреше; оператори Фредгольма; міри некомпактності

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О применении меры некомпактности в пространствах Фреше

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В метрических и топологических векторных пространствах понятие меры некомпактности используется для соответствия числовых значений множествам так, что компактные множества получили нулевые меры, а другие получили положительные значения, которые показывают, насколько они отличаются от компактных. Эта концепция была инициирована Куратовским в начале 30-х годов, и была определена и разработана многими различными способами. Меры некомпактности могут дать нам достаточные условия для формулировки различных теорем о неподвижных точках в метрических пространствах. Другое важное применение этих мер заключается в характеризации операторов Фредгольма в бесконечномерных топологических векторных пространствах. Целью данной работы является создание соответствующего критерия, который устанавливает связь между операторами Липшица-Фредгольма в более общем контексте пространств Фреше и меры некомпактности Хаусдорфа. Кроме того, используя произвольную меру некомпактности в смысле Банаса и Гебеля, мы получаем теорему о неподвижной точке для пространств Фреше.

Ключевые слова: пространства Фреше; операторы Фредгольма; меры некомпактности

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