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An alternative solution to the first boundary value problem for a differential equation of hyperbolic type

The construction of an approximate solution of the first boundary value problem for a hyperbolic type of differential equation using trigonometric splines is described. The expression of the approximate solution in trigonometric splines is proposed as an alternative solution, the minimization of the discrepancy is carried out by the method of collocations.

On a rectangular domain $D=[0,\pi]\times[0,\pi]$ we consider a hyperbolic differential equation:

$$\frac{\partial^2 u}{\partial y^2}-g(x,y)\cdot\frac{\partial^2 u}{\partial x^2}=f(x,y), \qquad (1)$$

where $f(x,y)$ and $g(x,y)$ are some continuous functions of two variables x and y.

Let's find a function of two variables $u(x,y)$, which is twice continuously differentiable with respect to both arguments in the domain D. Function $u(x,y)$ satisfies the equation (1) in the domain D and boundary conditions on the boundary of D:

$$u(0,y)=F_1(y),\ u(\pi,y)=F_2(y),\ u(0,x)=F_3(x), \qquad (2)$$

where $F_1(y)$, $F_2(y)$ and $F_3(x)$ are the given functions.

On the domain D we set a grid $\{x_i,y_j\}_{i=1,j=0}^{N-1,M-1}$, where $x_i=ih_x$, $y_j=jh_y$, $h_x=\dfrac{\pi}{N}$, $h_y=\dfrac{\pi}{M}$; and N, M are odd natural numbers, such that $N\geq 2$, $M\geq 2$; h_x, h_y are steps of the grip in x and y, respectively.

We will look for the solution of problem (1)-(2) in the form of a linear combination of the product of linearly independent and twice continuously differentiable functions $S(k,t)$ and $S(l,v)$, where $k=0,1,...,N-1$, $l=1,2,...,M-1$,

$$u(x,y)=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}a_{kl}\cdot S(k,t)S(l,v), \qquad (3)$$

$$S(k,t)=\frac{2}{N-1}\cdot\left[\frac{1}{2}+\sum_{i=1}^{N-2}\varphi(it)\cdot\cos(ix_k)+\frac{1}{2}\varphi((N-1)t)\cdot\cos((N-1)x_k)\right],$$

$$\varphi(i,t)=\frac{\frac{\cos(it)}{i^4}+\sum_{m=1}^{100}(\frac{\cos((2mN+i)t)}{(2mN+i)^4}+\frac{\cos((2mN-i)t)}{(2mN-i)^4})}{\frac{1}{i^4}+\sum_{m=1}^{100}\left(\frac{1}{(2mN+i)^4}+\frac{1}{(2mN-i)^4}\right)},$$

where a_{kl} are unknown coefficients.

Let the boundary conditions (2) be satisfied. In the nodes of the grid on the boundary $x=0$, we will accept:

$$\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}a_{kl}\cdot S(k,0)S(l,y_j)=F_1(y_j),\quad j=0,1,...,M-1. \tag{4}$$

Under conditions $S(k,0)=2$ for $k=0$ and $S(k,0)=0$ for $k\neq 0,$ the system of equations (4) has the form:

$$\sum_{i=0}^{M-1}2a_{0l}\cdot S(l,y_j)=F_1(y_j),\quad j=0,1,...,M-1.$$

Solving this system, we will find coefficients a_{0l} for $l=0,1,...,M-1.$

In the nodes of the grid on the boundary $x=\pi,$ we will get

$$\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}a_{kl}\cdot S(k,\pi)S(l,y_j)=F_2(y_j),\quad j=0,1,...,M-1. \tag{5}$$

Taking into account the conditions $S(k,\pi)=2$ for $k=N-1$ and $S(k,\pi)=0$ for $k\neq N-1,$ the system of equations (5) takes the form

$$\sum_{j=0}^{M-1}2a_{N-1l}\cdot S(l,y_j)=F_2(y_j).$$

Solving the resulting system, we will find the unknown coefficients a_{N-1l} for $l=0,1,...,M-1.$

At the nodal points on the boundary $y=0$, we get:

$$\sum_{k=1}^{N-2}\sum_{l=0}^{M-1}a_{kl}\cdot S(k,x_i)S(l,0)=F_3(x_i),$$

$$\sum_{k=1}^{N-2}a_{k0}\cdot S(k,x_i)=\frac{F_3(x_i)}{2},\quad i=1,2,\ ...,N-2.$$

The solutions of the obtained system of equations are coefficients a_{k0}, $k=1,2,\ ...,N-2.$

Then, we find second partial derivatives u''_{xx} and u''_{yy}:

$$u''_{xx}=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}S''_{tt}(k,t)\cdot S(l,v),\ \ u''_{yy}=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}S(k,t)\cdot S''_{vv}(l,v), \tag{6}$$

where

$$S''_{vv}(l,v)=\frac{2}{M-1}\left[\sum_{j=1}^{M-2}\varphi''(jv)\cdot\cos(jy_l)-\frac{1}{2}\varphi''((M-1)v)\cdot\cos((M-1)y_l)\right],$$

$$S''_{tt}(k,t)=\frac{2}{N-1}\left[\sum_{j=1}^{N-2}\varphi''(it)\cdot\cos(ix_k)-\frac{1}{2}\varphi''((N-1)t)\cdot\cos((N-1)x_k)\right],$$

$$\varphi''(j,v)=\frac{-\frac{\cos(jv)}{j^2}-\sum_{m=1}^{100}\left(\frac{\cos((2mM+j)v)}{(2mM+j)^2}+\frac{\cos((2mM-j)v)}{(2mM-j)^2}\right)}{\frac{1}{j^4}+\sum_{m=1}^{100}\left(\frac{1}{(2mM+j)^4}+\frac{1}{(2mM-j)^4}\right)},$$

$$\varphi''(i,t)=\frac{-\frac{\cos(it)}{i^2}-\sum_{m=1}^{100}\left(\frac{\cos((2mM+i)t)}{(2mN+i)^2}+\frac{\cos((2mM-i)t)}{(2mN-i)^2}\right)}{\frac{1}{i^4}+\sum_{m=1}^{100}\left(\frac{1}{(2mN+i)^4}+\frac{1}{(2mN-i)^4}\right)}.$$

Substituting expressions (3) and (6) into the equation (1), we obtain:

$$\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}a_{kl}\left[S(k,t)\cdot S''_{vv}(l,v)-g^2(t,v)\cdot S''_{tt}(k,t)\cdot S(l,v)\right]=f(t,v). \tag{7}$$

We put $b(k,l,t,v)=S(k,t)\cdot S''_{vv}(l,v)-g^2(t,v)\cdot S''_{tt}(k,t)\cdot S(l,v).$

Then, the system of equations (7) has the form:

$$\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}a_{kl}\cdot b(k,l,t,v)=f(t,v).$$

Finding the coefficients a_{0l}, a_{N-1l}, $l=0,1,...,M-1$, and a_{k0}, $k=1,2,...,N-2$, we will have:

$$\sum_{k=1}^{N-2}\sum_{l=1}^{M-1}a_{kl}\cdot b(k,l,t,v)=f(t,v)-\sum_{l=0}^{M-1}\left[a_{0l}b(0,l,t,v)+a_{N-1l}b(N-1,l,t,v)\right]-$$
$$-\sum_{k=1}^{N-2}a_{k0}b(k,0,t,v). \tag{8}$$

We minimize the disconnection

$$\varepsilon=\sum_{k=1}^{N-2}\sum_{l=1}^{M-1}a_{kl}\cdot b(k,l,t,v)-f(t,v)+\sum_{l=0}^{M-1}\left[a_{0l}b(0,l,t,v)+a_{N-1l}b(N-1,l,t,v)\right]+$$

$+\sum_{k=1}^{N-2}a_{k0}b(k,0,t,v)$ by the method of collocations and equate the disconnection in the grid nodes to zero. Then, we get a system of the equations:

$$\sum_{k=1}^{N-2}\sum_{l=1}^{M-1}a_{kl}\cdot b(k,l,x_i,y_j)-f(x_i,y_j)+\sum_{l=0}^{M-1}\left[a_{0l}b(0,l,x_i,y_j)+a_{N-1l}b(N-1,l,x_i,y_j)\right]$$

$$+\sum_{k=1}^{N-2} a_{k0} b(k,0,x_i,y_j) = 0.$$

Therefore,

$$\sum_{k=1}^{N-2}\sum_{l=1}^{M-1} a_{kl} \cdot b(k,l,x_i,y_j) = f(x_i,y_j) - \sum_{l=0}^{M-1}\left[a_{0l} b(0,l,x_i,y_j) + a_{N-1l} b(N-1,l,x_i,y_j)\right]$$

$$-\sum_{k=1}^{N-2} a_{k0} b(k,0,x_i,y_j),\ \ i = 1,2,\ldots,N-2,\ \ j = 0,1,\ldots,M-1.$$

Finding the unknown coefficients and putting them into the system (8), we will obtain an approximation of the solution of the boundary value problem (1)-(2).

Conclusions

Constructing an approximate solution by using paired trigonometric splines is proposed as an alternative approach to solving the first boundary value problem for a hyperbolic type of differential equation. It is worth noting that the convergence of approximate partial solutions of the differential equations of the hyperbolic type constructed using approximation by trigonometric splines requires further research.

References

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