MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National aviation university

I.O. Lastivka, Y.G.Liashenko , V.K.Repeta,

HIGHER MATHEMATICS

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

A guide
to independent work of higher education applicants
of technical specialties

Kyiv 2023

UDC

Compilers:
I.O. Lastivka – Doctor of Engineering, Professor, Head of the Department of Higher Mathematics

Y.G.Liashenko – PhD in Physics and Mathematics, Associate Professor

V.K.Repeta – PhD in Physics and Mathematics, Associate Professor

Reviewer: PhD in Pedagogy, Associate Professor V.I.Trofymenko

Approved by the Methodological and Editorial Board of the National Aviation University (Minutes № ___ of ________ 2023).

Higher mathematics. Theory of functions of complex variable: A guide to independent work of higher education applicants of technical specialties/compil.: I.O. Lastivka, Y.G.Liashenko, V.K.Repeta.– К. : NAU, 2023. –40p.

The guide is compiled according to the syllabus of the "Higher Mathematics" subject. The guide includes examples of solutions of typical problems of the topic "Theory of functions of complex variable", questions and tasks for self-test and problems for self-assignments.

For higher education applicants of technical specialties.

CONTENTS

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INTRODUCTION

Independent work of the student is the main way of mastering the educational material during the time free from the compulsory classroom classes.

The purpose of independent work - deepening, generalization and consolidation of theoretical knowledge and practical skills of students in the "Higher Mathematics" subject by developing the ability to work independently with the academic literature.

Independent work of higher education applicants is carried out in the form of preparation for lectures and practical classes, performance of individual homework and performance of module tests. Such training involves independent study of theoretical material on each topic presented in the recommended literature and lecture notes. It is important to pay attention to the need for a clear assimilation of basic terms and definitions, understanding of their content, obligatory analysis of the use of theoretical information for the proposed tasks

The guide for independent work of higher-education applicants is compiled in accordance with the curricula of the course "Higher mathematics" for students of technical specialties.

The proposed methodical work presents tasks for independent and individual work. A significant number of tasks for independent work has an applied orientation.

The leading teacher can adjust the number and content of the tasks, which a student must perform independently while studying the relevant material.

The material of each topic corresponds to the working curricula of the "Higher mathematics" subject, in particular, to one of its sections "Number series". Each topic contains the basic methodical recommendations, recommended literature, typical examples of solutions and tasks for individual performance, and questions for self-checking, which will contribute to better understanding, assimilation and possibility to apply the basic theoretical statements.

This guide is compiled for independent work of higher education applicants of technical specialties and focused on the theoretical and methodological support of the training process.

Topic1. COMPLEX NUMBERS AND OPERATIONS WITH THEM

Plan

1. Complex numbers; algebraic, trigonometric, exponential forms of representation; geometrical interpretation.
2. Operations with complex numbers.
3. Euler's, de Moivre's formulas, extraction of *n*-th power root of complex number.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 1 the student should ***know***: definition of complex number, geometrical, forms of representation of complex number, Euler's, de Moivre's formulas, extraction of *n*-th power root of complex number; ***to be able to:*** perform operations with complex numbers, define modulus and argument of complex number, reduce complex number to *n*-th power and extract of *n*-th power root of complex number.

Basic theoretical information

Expression

$$z = x + iy,$$

where x and y are real numbers, $i^2 = -1$, i is imagine unit, is called *complex number*.

Number $x = \operatorname{Re} z$ is *real part* of number *z*; $y = \operatorname{Im} z$ is *imagine part* of number z.

Geometrically complex number $z=x+iy$ can be represented at the *xy*-plane as point with components x and y or radius-vector of point $A(x; y)$ (Fig. 1.1).

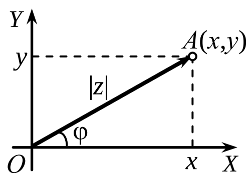


Fig. 1.1

The plane which points represent complex numbers is called *complex plane* and is denoted letter *C*. *x*-axis is called *real axis*, *y*-axis is called *imagine axis*.

Modulus of complex number $z = x + iy$ is a length of vector $\overrightarrow{OA}$ and is calculated by the formula: $|z| = \sqrt{x^2 + y^2}$.

Numbers $z=x+iy$ and $\overline{z}=x-iy$ are called *conjugate*.

The following equalities are true: $z\overline{z}=x^2+y^2=|z|^2=|\overline{z}|^2$.

An argument of number z is angle φ = Arg z to which positive part of x-axis should be turned around origin up to coinciding it to positive direction of vector $\overrightarrow{OA}$. An angle φ is positive if such turning is performed counterclockwise and angle φ is negative if the turning is performed clockwise.

If $z\neq 0$ then

$$\operatorname{Arg} z=\arg z+2\pi k,$$

where $\arg z\in(-\pi;\pi]$ is known to be a *principal* value of argument of complex number z, $\operatorname{Arg} z$ is known to be *general* value of argument of z; here k = 0, ±1, ±2, ±3,… .

The representation forms of complex number z (an expression of right part) are following:

$z=x+iy$ is algebraic form;

$z=|z|(\cos\varphi+i\sin\varphi)$ is trigonometric form;

$z=|z|e^{i\varphi}$ is exponential form.

The equation

$$e^{i\varphi}=\cos\varphi+i\sin\varphi$$

which defines the symbol $e^{i\varphi}$ for any real value of φ, is known as E*uler's formula*.

De Moivre's formula has the form:

$$z^n=|z|^n(\cos n\varphi+i\sin n\varphi) \tag{1.1}$$

The *formula of extraction of n-th power root* of complex number z:

$$\sqrt[n]{z}=\sqrt[n]{|z|}\left(\cos\frac{\varphi+2\pi k}{n}+i\sin\frac{\varphi+2\pi k}{n}\right), \tag{1.2}$$

where φ = arg z, $\sqrt[n]{|z|}$ is arithmetic value of root, $k=0;1;...;n-1$.

Examples of solution of typical problems

Example 1. Find the real and imagine parts of complex numbers:

a) $z_1=\dfrac{1}{i}$; b) $z_2=i^{31}$; c) $z_3=(2i-1)(3i+2)$; d) $z_4=\dfrac{i+3}{4-3i}$.

Solution: a) $z_1=\dfrac{1}{i}=\dfrac{i}{i^2}=\dfrac{i}{-1}=-i=0+(-1)i \Rightarrow \operatorname{Re} z_1=0, \operatorname{Im} z_1=-1$;

b) $z_2=i^{33}=i^{32}i=\left(i^2\right)^{16}\cdot i=(-1)^{16}\cdot i=i \Rightarrow \operatorname{Re} z_2=0, \operatorname{Im} z_2=1$;

c) $z_3=(2i-1)(3i+2)=2i\cdot 3i+4i-3i-2=6i^2+i-2=$
$=-6+i-2=-8+i \Rightarrow \operatorname{Re} z_3=-8, \operatorname{Im} z_3=1$;

d) $z_4=\dfrac{i+3}{4-3i}=\dfrac{(i+3)(4+3i)}{(4-3i)(4+3i)}=\dfrac{4i+12+3i^2+9i}{16-9i^2}=$
$=\dfrac{4i+12-3+9i}{16+9}=\dfrac{9+13i}{25}=\dfrac{9}{25}+\dfrac{13}{25}i \Rightarrow \operatorname{Re} z_4=\dfrac{9}{25}, \operatorname{Im} z_4=\dfrac{13}{25}$.

Remark. During division of complex number z_1 to number z_2 the transformation is used: $\dfrac{z_1}{z_2}=\dfrac{z_1\bar{z}_2}{z_2\bar{z}_2}$; ($z_2\neq 0$).

Example 2. Solve the equation $z^2-4z+5=0$.

Solution. The quadratic equation has negative discriminant: $D=16-4\cdot 5=-4=(2i)^2$. Hence, the equation has pare of complex-conjugate solutions:

$$z_1=\frac{4+2i}{2}=2+i;\quad z_2=\frac{4-2i}{2}=2-i.$$

Example 3. Find modulus and principal values of arguments of complex numbers and represent them in trigonometric form:

a) $z_1=-2$; b) $z_2=1-\sqrt{3}i$; c) $z_3=3i$; d) $z_4=4+3i$.

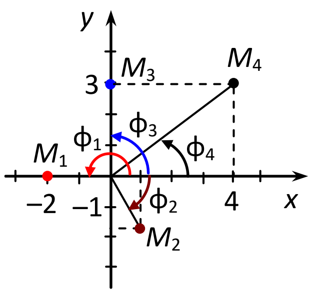


Fig. 1.2

Solution: a) z_1 is real and negative number. Its modulus equals to the distance from the point $M_1(-1;\,0)$ to the origin of complex plane (Fig.1.2): $|z_1|=2$,

$\arg z_1 = \varphi_1 = \pi$, $z_1 = \cos\pi + i\sin\pi$;

b) $|z_2| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2$, $\cos\varphi_2 = \frac{1}{2}$, $\sin\varphi_2 = -\frac{\sqrt{3}}{2}$

$\Rightarrow \varphi_2 = -\frac{\pi}{3} \in (-\pi;\pi]$.

Hence, $z_2 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$;

c) $z_3 = 3i$ is pure imagine number, its modulus is the distance from the point $M_3(0;3)$ to the origin: $|z_3| = 3$; $\arg z_3 = \frac{\pi}{2}$, then

$z_3 = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$;

d) similarly, $|z_4| = \sqrt{4^2+3^2} = 5, \cos\varphi_4 = \frac{4}{5}$, $\sin\varphi_4 = \frac{3}{5}, \varphi_4 = \operatorname{arctg}\frac{3}{4}$, then $z_4 = 5\left(\cos\varphi_4 + i\sin\varphi_4\right)$.

Example 4. Solve the equation $z^4 + 16 = 0$.

Solution. Let's transform the equation in following way: $z^4 = -16$ and represent number -16 in trigonometric form: $-16 = 16(\cos\pi + i\sin\pi)$.

According to (1.2) we get

$$\sqrt[4]{-16} = \sqrt[4]{16}\cdot\left(\cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}\right),\ k = 0;\ 1;\ 2;\ 3.$$

Successively we define all four roots of given equation:

$$k=0:\ z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2};$$

$$k=1:\ z_2 = 2\left(\cos\frac{\pi+2\pi}{4} + i\sin\frac{\pi+2\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2};$$

$$k=2:\ z_3 = 2\left(\cos\frac{\pi+4\pi}{4} + i\sin\frac{\pi+4\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2};$$

$$k=3:\ z_4=2\left(\cos\frac{\pi+6\pi}{4}+i\sin\frac{\pi+6\pi}{4}\right)=2\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=\sqrt{2}-i\sqrt{2}\ .$$

Geometrically points $M_k(x_k;y_k)$, which correspond obtained roots $z_k=x_k+i\,y_k$ are vertices of square, inscribed in a circle of radius 2 with the center at the origin (Fig. 1.3).

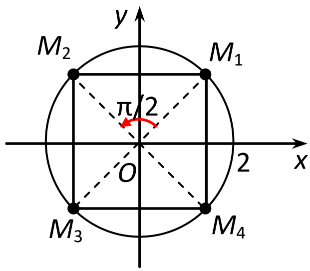


Fig. 1.3

Self-test questions

1. What is called an imaginary unit? What is a complex number?
2. How do you denote a complex number geometrically?
3. What are the modulus and argument of a complex number?
4. How to write a complex number in trigonometric and exponential forms?
5. By what rules the arithmetic operations with complex numbers are performed ?
6. Write Euler's formula; de Moivre’s formula; formula of extracting the root of the *n*-th power from a complex number *z*.

Self-test assignments

Task 1. Find the real and imagine parts of complex numbers:

a) $z=i^{45}+\frac{1}{i^{41}}$; b) $z=(4-i)(2+5i)+\frac{(1-i)^2}{1+i}$.

Task 2. Represent the complex numbers in trigonometric form:

a) $z=4$; b) $z=-1+i$; c) $z=-\sqrt{3}-i$; d) $z=-i$; e) $z=\sqrt{3}-\sqrt{6}i$.

Task 3. Calculate by the de Moivre’s formula:

a) $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{15}$; b) $\left(-\sqrt{3}+i\right)^{12}$.

Task 4. Solve the equations: a) $z^2-6z+13=0$; b) $z^2=4i$; c) $z^3=-8$.

Answers: **1.** a) $\operatorname{Re} z = 0, \operatorname{Im} z = 0$; b) $\operatorname{Re} z = 12, \operatorname{Im} z = 17$. **2.** a) $z = 4(\cos 0 + $ $+ i \sin 0)$; b) $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$; c) $z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$; d) $z = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$; e) $z = 3(\cos\varphi + i\sin\varphi)$, де $\varphi = \operatorname{arctg}\left(-\sqrt{2}\right)$. **3.** a) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$; b) 4096. **4.** a) $3+2i$, $3-2i$. b) $\sqrt{2}+\sqrt{2}i$, $-\sqrt{2}-\sqrt{2}i$; c) $1+\sqrt{3}i$, –2; $1-\sqrt{3}i$.

Topic 2. FUNCTIONS OF COMPLEX VARIABLE

Plan

1. Definition of the function of a complex variable.
2. The basic domains in the complex plane.
3. Series with complex terms.
4. Basic elementary functions.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 2 the student should ***know:*** definition of function of complex variable, main curves and domains in complex plane, definition of basic elementary functions and their properties; ***be able to:*** separate the real and imaginary parts of the function of a complex variable, examine series with complex terms for convergence, represent lines and domains given by equations and inequalities in the complex plane.

The basic theoretical information

Definition of the function of a complex variable. If one or more complex numbers $w \in E \subset C$ are matched to each $z \in D \subset C$ according to a certain law, then it is said that a *function of a complex variable* is defined on the set D, and it is denoted $w = f(z)$.

If only one number w corresponds to each $z \in D$ then the function $w = f(z)$ is called *single-valued*, and if there is more than one value of w then it is called *multi-valued*.

Function $w = f(z)$ can be written in the form

$$w = f(z) = u(x, y) + iv(x, y),$$

where $u = u(x,y)$ and $v = v(x,y)$ are real functions of real arguments x and y.

The basic domains in the complex plane. The set D of the complex plane is called a *domain* if it is *open* and *connected*, i.e.:

1) together with each point of set D, this set also completely contains a certain circle centered at this point;

2) any two points of the set D can be connected by a continuous line that lies entirely in D.

A domain D is called *simply connected* if along with any continuous closed self-nonintersecting curve drawn in D, the domain D also includes the entire domain bounded by this curve.

Examples of simply connected domains: circle $|z - z_0| < R$, the whole complex plane z, semiplanes $\operatorname{Re} z > 0$ and $\operatorname{Im} z < 0$ etc.

Note that the annulus $r < |z - z_0| < R$ where $0 \le r < R$ is a double connected domain.

Neighborhood (δ-neighborhood) of a point is a circle $|z - z_0| < \delta$ centered at point z_0 and with radius δ. In general neighborhood is known to be any domain included the point z_0.

Series with complex terms. Expression of the form

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \ldots + z_n + \ldots ,$$

is called a *number* series (in the complex domain), here $z_n = x_n + iy_n$ ($n = 1, 2, \ldots$) are complex numbers.

Power series is the series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots ,$$

where $a_0, a_1, a_2, \ldots$ are complex coefficients of the power series; $z = x + iy$ is complex variable, z_0 is fixed complex number.

If $z_0 = 0$ power series acquires the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \qquad (2.1)$$

The set of all values z for which the series (2.1) converges is called its domain of convergence.

Formulas for determination of radius of convergence of power series: $R = \lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$ або $R = \dfrac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}$.

Basic elementary functions and their properties:

$e^z = 1 + z + \dfrac{z^2}{2!} + \ldots + \dfrac{z^n}{n!} + \ldots;$

$\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots + (-1)^n \dfrac{z^{2n}}{(2n)!} + \ldots;$

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots + (-1)^n \dfrac{z^{2n+1}}{(2n+1)!} + \ldots;$

$e^{iz} = \cos z + i\sin z$ is Euler's formula;

e^z is periodic function with period $2\pi i$;

$\cos z = \dfrac{1}{2}\left(e^{iz} + e^{-iz}\right)$, $\sin z = \dfrac{1}{2i}\left(e^{iz} - e^{-iz}\right)$;

$\operatorname{ch} z = \dfrac{1}{2}\left(e^{z} + e^{-z}\right)$, $\operatorname{sh} z = \dfrac{1}{2}\left(e^{z} - e^{-z}\right)$ are hyperbolic cosine and sine respectively;

$\sin iz = i\operatorname{sh} z$, $\cos iz = \operatorname{ch} z$, $\operatorname{sh} iz = i\sin z$, $\operatorname{ch} iz = \cos z$;

$\operatorname{Ln} z = \ln|z| + i\left(\arg z + 2\pi k\right)$, $k \in Z$;

$a^z = e^{z\operatorname{Ln} a}$ is general exponent function;

z^n, $n \in N$ is power function;

$z^\alpha = e^{z\operatorname{Ln}\alpha}$ is general power function;

$$\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln}\left(iz + \sqrt{1-z^2}\right),\ \operatorname{Arccos} z = \frac{1}{i}\operatorname{Ln}\left(z + \sqrt{z^2-1}\right),$$

$$\operatorname{Arctg} z = \frac{1}{2i}\operatorname{Ln}\frac{1+iz}{1-iz}\,(z \neq \pm i),\ \operatorname{Arcctg} z = \frac{1}{2i}\operatorname{Ln}\frac{iz-1}{iz+1}\,(z \neq \pm i).$$

Examples of solution of typical problems

Example 1. Write the function $w = f(z)$ in the form $w = u(x,y) + iv(x,y)$:

a) $w = 2z - i\overline{z}$; b) $w = z^3$; c) $w = e^{-z}$.

Solution. a) $w = 2z - i\overline{z} = 2(x+iy) - i(x-iy) = \underbrace{2x-y}_{u(x,y)} + i\underbrace{(2y-x)}_{v(x,y)}$;

b) $w = z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3 =$

$= x^3 + 3x^2iy - 3xy^2 - iy^3 = \underbrace{x^3 - 3xy^2}_{u(x,y)} + i\underbrace{\left(3x^2y - y^3\right)}_{v(x,y)}$;

c) $w = e^{-z} = e^{-x-iy} = e^{-x}e^{-iy} = e^{-x}(\cos y - i\sin y) =$

$= \underbrace{e^{-x}\cos y}_{u(x,y)} + i\underbrace{\left(-e^{-x}\sin y\right)}_{v(x,y)}$.

Example 2. Define the lines, which are given by the equations:

a) $|z+i| = 2$; b) $\arg z = \pi$; c) $z = \cos t + i\sin t,\ t \in [0;\pi]$;

d) $z = z_0 + Re^{it}, t \in [0; 2\pi]$.

Solution. a) we have the equation of the circle with radius 2 centered at point $(0;-1)$ (Fig. 2.1, a) Really,

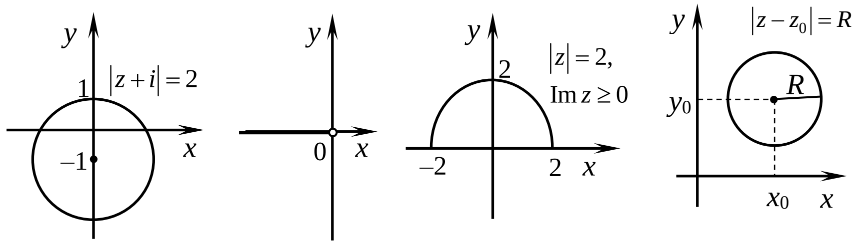
$$|z+i| = 2 \Leftrightarrow |x+iy+i| = 2 \Leftrightarrow \sqrt{x^2+(y+1)^2} = 2 \Leftrightarrow x^2 + (y+1)^2 = 4;$$

b) all points of complex plane having the principal value of argument equaled to π form negative semiaxis of x-axis except point (0;0) (Fig.2.1,b);

c) if $z = x + iy = \cos t + i\sin t, t \in [0;\pi]$, then $x = \cos t, y = \sin t, \quad y \ge 0$. Then $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ (Fig. 2.1, c);

d) $z = z_0 + Re^{it}, t \in [0;2\pi] \Leftrightarrow \begin{cases} x - x_0 = R\cos t, \\ y - y_0 = R\sin t, \end{cases} t \in [0;2\pi]$.

The parametric equations of the circle $(x - x_0)^2 + (y - y_0)^2 = R^2$ have been obtained. (Fig. 2.1, d).



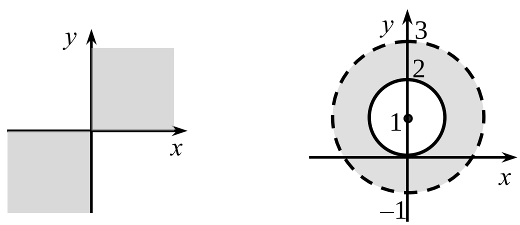
a *b* *c* *d*

Fig. 2.1

Example 3. Determine sets of points on the complex plane z which are defined by inequalities: a) $\operatorname{Im} z^2 \ge 0$; b) $1 \le |z - i| < 2$.

Solution: a) $z^2 = (x + iy)^2 = x^2 - y^2 + (2xy)i \Rightarrow \operatorname{Im} z^2 = 2xy$. Hence, $xy \ge 0$. This inequality defines the set of all points, located in the first and third quadrants (Fig.2.2, a);

b) desired set of points is a ring bounded by circles $|z - i| = 1$ and $|z - i| = 2$ (Fig. 2.2, б).



a *б*

Fig. 2.2

Example 4. Find the domain of convergence of the series $\sum_{n=1}^{\infty}\frac{n}{(z-i)^n}$.

Solution. According to D'Alembert's test we obtain

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{n+1}{(z-i)^{n+1}}\cdot\frac{(z-i)^n}{n}\right|=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{|z-i|}=\frac{1}{|z-i|}<1.$$

Hence, it follows for any z satisfying inequality $|z-i|>1$ (the outer part of a circle of radius 1 centered at a point $(0;1)$), the series is absolutely convergent. For points z of a circle $|z-i|=1$ we have a series $\sum_{n=1}^{\infty}\frac{n}{e^{in\varphi}}=\sum_{n=1}^{\infty}ne^{-in\varphi}=\sum_{n=1}^{\infty}n\cos n\varphi-i\sum_{n=1}^{\infty}n\sin n\varphi$. Since both series with real terms divergent, the series $\sum_{n=1}^{\infty}\frac{n}{e^{in\varphi}}$ is also divergent. So, domain of convergence of this series is given by the inequality $|z-i|>1$.

Example 5. Write the expressions in algebraic form:

a) $\sin\left(\frac{\pi}{3}+2i\right)$; b) $\mathrm{Ln}\left(\sqrt{3}-i\right)$.

Solution: a) $\mathrm{w}=\sin\left(\frac{\pi}{3}+2i\right)=\sin\frac{\pi}{3}\cos 2i+\cos\frac{\pi}{3}\sin 2i=$

$$=\frac{\sqrt{3}}{2}\mathrm{ch}\,2+\frac{1}{2}(\mathrm{sh}\,2)i\Rightarrow \mathrm{Re}\,w=\frac{\sqrt{3}}{2}\mathrm{ch}\,2,\mathrm{Im}\,z=\frac{1}{2}\mathrm{sh}\,2\ ;$$

b) $\mathrm{Ln}\left(\sqrt{3}-i\right)=\ln\left|\sqrt{3}-i\right|+i\left(\arg\left(\sqrt{3}-i\right)+2\pi k\right),\ k\in Z$,

$\left|\sqrt{3}-i\right|=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=\sqrt{3+1}=2$, $\arg\left(\sqrt{3}-i\right)=\mathrm{arctg}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$.

Hence, $\mathrm{Ln}\left(\sqrt{3}-i\right)=\ln 2+i\left(-\frac{\pi}{6}+2\pi k\right),\ k\in Z$;

Self-test questions

1. Give the definition of a function of a complex variable.
2. What domain is called simply connected?

3. What is a power series?

4. How to find the radius of convergence of a power series?

5. Define the functions $w=e^z$, $w=\sin z$, $w=\cos z$, $w=shz$, $w=chz$, $w=Lnz$, $w=Arc\cos z$, $w=a^z$, $w=z^a$.

6. What is the relationship between trigonometric and hyperbolic functions?

Self-test assignments

Task 1. Write the function $w=f(z)$ as $w=u(x,y)+iv(x,y)$:

a) $w=\overline{z}^2+2iz$; b) $w=\dfrac{1}{z}$; c) $w=ze^z$.

Task 2. Define the lines given by the equations:

a) $|z-1-2i|=3$; b) $\arg z=-\dfrac{\pi}{4}$; c) $z=2\cos t+3i\sin t,\ t\in[0;2\pi]$;

d) $z=Re^{it},\ t\in\left[-\dfrac{\pi}{2};\dfrac{\pi}{2}\right]$.

Task 3. Define sets of points on the complex plane z which are determined by the inequalities: a) $\operatorname{Re} z<0$; b) $\begin{cases}|z+1|<2,\\ \arg z\ge\dfrac{\pi}{2}.\end{cases}$

Task 4. Investigate the series $\sum\limits_{n=1}^{\infty}\dfrac{e^{2in}}{n}$ on convergence.

Task 5. Define the domain of convergence of the series $\sum\limits_{n=1}^{\infty}\dfrac{(z+2i)^n}{n^2+1}$.

Task 6. Write the following expressions in algebraic form:

a) $\cos\left(\dfrac{\pi}{4}-3i\right)$; b) $\operatorname{Ln}(-1)$; c) 1^i; d) $\operatorname{sh}(i-2)$; e) $\operatorname{tg}2i+\operatorname{th}2i$.

Answers: **1.** a) $w=x^2-y^2-2y+2x(1-y)i$; b) $\dfrac{x}{x^2+y^2}-\dfrac{yi}{x^2+y^2}$; c) $w=e^x\left(x\cos y-y\sin y\right)+ie^x\left(x\sin y+y\cos y\right)$. **2.** a) circle with radius 3 centered at point (1; 2); b) a ray with an excluded origin (0; 0) that bisects the fourth quarter; c) ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$; d) a semicircle centered at the point (0; 0) with radius 1, located in the I and IV quarters. **3**. a) half-plane located to the left of the ordinate axis; b) a part of a circle with radius 2 and centered at the point (–1;0) (without boundary), located in the second coordinate quarter, including the corresponding points of the coordinate axes. **4.** Divergent. **5.** Circle (with boundary) with radius 1 centered at point (0; –2). **6.** a) $\frac{\operatorname{ch}3}{\sqrt{2}}+\frac{\operatorname{sh}3}{\sqrt{2}}i$; b) $\pi(2k+1)i, k\in Z$; c) $e^{-2\pi k}$, $k\in Z$; d) $-\operatorname{sh}2\cos 1+(\operatorname{ch}2\sin 1)i$; e) $(\operatorname{tg}2+\operatorname{th}2)i$.

Topic 3. DIFFERENTIATION OF FUNCTIONS OF A COMPLEX VARIABLE

Plan

1. Definition of derivative.
2. Cauchy–Riemann's conditions.
3. Analytical functions.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 3 the student should ***know***: definition of derivative, properties of derivatives, Cauchy–Riemann's conditions, definition of analytic and harmonic functions; ***be able to***: examine functions for differentiability, reproduce an analytical function by its known real or imaginary parts.

The basic theoretical information

Definition of a derivative. Let a single-valued function $f(z)$ be defined in the domain D and $z\in D$.

The *derivative* of the function $f(z)$ at the point z is the limit of the ratio of the increment of the function $f(z)$ at the point z to the increment of the argument Δz, if the increment of the argument approaches to zero, i.e.

$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}.$$

A function $f(z)$ which has a finite derivative $f'(z)$ at a point $z \in D$ is called *differentiated* at this point. A function differentiated at each point of the domain is called differentiated in this domain.

Cauchy–Riemann's conditions. A function $f(z) = u(x, y) + iv(x, y)$ is differentiated at a point $z = x + iy$ if and only if the real functions $u(x, y)$ and $v(x, y)$ are differentiated at the point (x, y) and the Cauchy–Riemann's (or Euler–D'Alembert's) conditions are fulfilled:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \,. \tag{3.1}$$

Then the formulas are true:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \,. \tag{3.2}$$

Analytical functions. A single-valued function $f(z)$ is called *analytic* at point z if it is differentiated in some neighborhood of this point. The function *f(z)* is called *analytic* in the domain D if it is differentiated at each point of this domain.

The real function $\varphi(x, y)$, which has continuous partial derivatives up to the second order at the domain D and satisfies the Laplace equation $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$, is called a *harmonic* function at this domain.

Formulas for reconstitution an analytic function $f(z) = u(x, y) + iv(x, y)$ by known harmonic real $u(x, y)$ or imaginary $v(x, y)$ part are following, respectively:

$$v(x, y) = \int_{x_0}^{x} \left(-\frac{\partial u(x, y_0)}{\partial y} \right) dx + \int_{y_0}^{y} \frac{\partial u(x, y)}{\partial x} dy + c \,;$$

$$u(x, y) = \int_{x_0}^{x} \frac{\partial v(x, y_0)}{\partial y} dx + \int_{y_0}^{y} \left(-\frac{\partial v(x, y)}{\partial x} \right) dy + c \,, \tag{3.3}$$

where c is arbitrary constant.

Examples of solution of typical problems

Example 1. Is the function differentiated $f(z) = z^2 = x^2 - y^2 + 2xyi$?

Solution: $u = x^2 - y^2$, $v = 2xy$, $\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y, \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x$

The Cauchy–Riemann's conditions (3.1) are satisfied. Therefore, the function is differentiated at each point of the complex plane. We find the derivative using the first formula (3.2):

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i \cdot 2y = 2z .$$

Example 2. Find the analytic function $f(z) = u(x, y) + iv(x, y)$ if

$$v = -2xy - 3x, \quad f(0) = i.$$

Solution. The function $v(x, y)$ is harmonic in the whole complex plane. Indeed, since $\frac{\partial v}{\partial x} = -2y - 3$, $\frac{\partial v}{\partial y} = -2x$, $\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = 0$ and $\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = 0$, then the function $v(x, y)$ satisfies the Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

According to formula (3.3), in which $x_0 = 0,\ y_0 = 0$, we get

$$u(x, y) = \int_0^x (-2x)dx + \int_0^y (2y + 3)dy + c = -x^2 + y^2 + 3y + c .$$

Hence, $f(z) = u(x, y) + iv(x, y) = -x^2 + y^2 + 3y + c + i\left(-2xy - 3x\right) =$

$$= -x^2 + y^2 - 2xyi + 3y - 3xi + c =$$

$$= -(x + iy)^2 - 3i(x + iy) + c = -z^2 - 3iz + c .$$

From the condition $f(0) = i$ the constant $c = i$ is determined. Finally, we have: $f(z) = -z^2 - 3iz + i$.

Self-test questions

1. Give the definition of derivative of function of a complex variable.
2. What are conditions of differentiability of function of a complex variable.
3. Give the definition of analytic function.

4. Give the definition of harmonic function.
5. How the analytic function can be reconstituted by given either real or imagine part?

Self-test assignments

Task 1. Check the function on differentiability $f(z)=x^2+y^2-2xyi$.

Task 2. Is the function $f(z)=\sin x \operatorname{ch} y + i\cos x\,sh y$ differentiated? If it is so, find its derivative.

Task 3. Find derivatives $z'(t)$ of functions:

a) $z(t)=\cos^2 t+ie^{-t}$; b) $z(t)=\ln(t^2+1)+i\,\operatorname{arctg}\frac{1}{t}$.

Task 4. Check the fulfillment of the Cauchy-Riemann's conditions and, if they are fulfilled, find $f'(z)$:

a) $f(z)=e^{3z}$; b) $f(z)=\operatorname{sh} z$; c) $f(z)=\cos z$.

Task 5. Define the analytic function $f(z)=u+iv$ if:

a) $u=x^2-y^2+2x$, $f(i)=-1+2i$; b) $u=\dfrac{x}{x^2+y^2}$, $f(\pi)=\dfrac{1}{\pi}$;

c) $v=\sin x \operatorname{sh} y$, $f(0)=0$.

Answers: 1. It is non-differentiated in the whole complex plane, except for the imaginary axis. **2.** Yes. $f'(z)=\cos z$. **3.** a) $z'(t)=-\sin 2t-i\,e^{-t}$; b) $z'(t)=\dfrac{2t}{t^2+1}-i\dfrac{1}{1+t^2}$. **4.** a) $3e^{3z}$; b) chz; c) $-\sin z$. **5.** a) $f(z)=z^2+2z$; b) $f(z)=\dfrac{1}{z}$; c) $f(z)=-\cos z+1$.

Topic 4. INTEGRATION OF FUNCTIONS OF A COMPLEX VARIABLE

Plan

1. Formulas for calculation of the integral of function of a complex variable.
2. Cauchy's integral theorem.
3. Newton-Leibnitz' formula.
4. Cauchy's integral formula.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 4 the student should ***know***: definition of the integral of function of a complex variable, formulation of Cauchy's integral theorem, Newton-Leibnitz' formula; **be able to**: integrate analytical and non-analytical functions of a complex variable.

The basic theoretical information

Formulas for calculation the integral of function of a complex variable. If the function $f(z)=u(x,y)+iv(x,y)$ is continuous on a smooth curves L, then the integral $\int_L f(z)dz$ exists and the following formula is true

$$\int_L f(z)dz = \int_L udx - \upsilon dy + i\int_L \upsilon dx + udy. \qquad (4.1)$$

If the functions u and υ are continuous along a smooth arc defined parametrically: $x = x(t)$, $y = y(t), t \in [\alpha;\beta]$, then

$$\int_L f(z)dz = \int_\alpha^\beta (u+i\upsilon)(x'+iy')dt = \int_\alpha^\beta f(z(t))z'(t)dt.$$

Cauchy's integral theorem. If the function $f(z)$ is analytic in the one-connected domain D and L is an arbitrary piecewise smooth closed contour lying entirely in D, then

$$\oint_L f(z)dz = 0.$$

The Newton–Leibnitz formula. If $f(z)$ is the analytic function in the one-connected domain D and *Φ(z)* is any antiderivative for $f(z)$, then

$$\int_{z_1}^{z_2} f(z)dz = \Phi(z_2) - \Phi(z_1).$$

Cauchy's integral formulas:

$$f(z_0) = \frac{1}{2\pi i}\oint_L \frac{f(z)dz}{z-z_0}.$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_L \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots).$$

Consequences from Cauchy's integral formulas:

$$\oint_L \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0), \qquad (4.2)$$

$$\oint_L \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0). \qquad (4.3)$$

Examples of solution of typical problems

Example 1. Calculate the integral $\int_L (2-i+3\overline{z})dz$ along lines connected the points $z_1 = 0$ i $z_2 = 1-i$ in direction from z_1 to z_2:

a) along the segment;

b) along the arc of the parabola $y = -x^2$;

c) broken line $z_1 z_3 z_2$ where $z_3 = 1$.

Solution. Let's rewrite the integrand function in the form

$$2-i+3\overline{z} = 2-i+3(x-iy) = \underset{u(x,y)}{2+3x} - i\underbrace{(1+3y)}_{v(x,y)}.$$

According the formula (4.1) we have

$$I = \int_L (2-i+3\overline{z})dz = \int_L (2+3x-i(1+3y))(dx+idy) =$$

$$= \int_L (2+3x)dx + (1+3y)dy + i\int_L (2+3x)dy - (1+3y)dx;$$

a) the equation of a segment connected points $z_1 = 0$ and $z_2 = 1-i$ has the form $y = -x$, $0 \le x \le 1$; hence $dy = -dx$.

Then

$$I = \int_0^1 \big((2+3x)+(1+3(-x))(-1)\big)dx + i\int_0^1 \big((2+3x)(-1)-(1+3(-x))\big)dx =$$

$$= \int_0^1 (1+6x)dx + i\int_0^1 (-3)dx = \left(x+3x^2\right)\Big|_0^1 - 3ix\Big|_0^1 = 1+3-3i = 4-3i;$$

b) for the parabola $y = x^2$ we have $dy = 2xdx$ $(0 \le x \le 1)$, hence,

$$I = \int_0^1 \Big((2+3x)+(1+3(-x^2))(-2x)\Big)dx +$$

$$+i\int_0^1\Big((2+3x)(-2x)-(1+3(-x^2))\Big)dx=$$

$$=\int_0^1\left(2+3x-2x+6x^3\right)dx+i\int_0^1\left(-4x-6x^2-1+3x^2\right)dx=$$

$$=\int_0^1\left(2+x+6x^3\right)dx+i\int_0^1\left(-4x-3x^2-1\right)dx=\left(2x+\frac{x^2}{2}+\frac{6x^4}{4}\right)\Bigg|_0^1+$$

$$+i\left(-2x^2-x^3-x\right)\Bigg|_0^1=2+\frac{1}{2}+\frac{3}{2}+i\left(-2-1-1\right)=4-4i\ ;$$

c) $I=I_1+I_2$, where I_1 is integral of given function along segment connecting points (0; 0) i (1; 0) (it is called as segment z_1z_3), respectively, I_2 is integral of given function along segment z_3z_2. On the segment z_1z_3: $y = 0$, $dy = 0$, $0 \le x \le 1$, then

$$I_1=\int_0^1(2+3x)dx+i\int_0^1(-(1+3\cdot 0))dx=$$

$$=\int_0^1(2+3x)dx-i\int_0^1 dx=\left(2x+\frac{3x^2}{2}-ix\right)\Bigg|_0^1=3{,}5-i\,.$$

On the segment z_3z_2: $x=1, dx=0, y\in[0;-1]$. Then

$$I_2=\int_0^{-1}(1+3y)dy+i\int_0^{-1}(2+3\cdot 1)dy=$$

$$=\left(y+\frac{3y^2}{2}+5iy\right)\Bigg|_0^{-1}=-1+\frac{3}{2}-5i=0{,}5-5i\,.$$

Hence, $I=I_1+I_2=3{,}5-i+0{,}5-5i=4-6i$.

We make sure of this example that value of integral of continuous but nonanalytic function depends on the form of the path of integration

Example 2. Evaluate the integral $\int_L(z^2+\overline{z}^2+4z\overline{z})dz$ if L is arc of circle $|z|=2$ $(0\le\arg z\le\pi)$.

Solution. Assume $z = 2e^{i\varphi}$, $dz = 2ie^{i\varphi}d\varphi$. Then

$$\int_L (z^2+\bar{z}^2+4z\bar{z})dz=\int_0^{\pi}\left(4e^{2i\varphi}+4e^{-2i\varphi}+4\cdot 2e^{i\varphi}\cdot 2e^{-i\varphi}\right)\cdot 2ie^{i\varphi}d\varphi=$$

$$=8i\int_0^{\pi}\left(e^{3i\varphi}+e^{-i\varphi}+4e^{i\varphi}\right)d\varphi=8i\left(\frac{e^{3i\varphi}}{3i}+\frac{e^{-i\varphi}}{-i}+\frac{4e^{i\varphi}}{i}\right)\Bigg|_0^{\pi}=$$

$$=8\left(\frac{e^{3i\varphi}}{3}-e^{-i\varphi}+4e^{i\varphi}\right)\Bigg|_0^{\pi}=8\left(\frac{e^{3i\pi}}{3}-e^{-i\pi}+4e^{i\pi}\right)-8\left(\frac{1}{3}-1+4\right)=$$

$$=8\left(\frac{-1}{3}+1-4\right)-8\left(\frac{1}{3}+3\right)=-8\left(\frac{2}{3}+6\right)=\frac{160}{3}=-53\frac{1}{3}.$$

Example 3. Evaluate the integral

$$\int_L \left(4z^3+\sin 2z\right)dz;\quad L:\{|\,z\,|=1,\ \operatorname{Re} z\geq 0\}\ .$$

Solution. Integrand is analytic function, consequently integral does not depend on line of integration and depends on initial and terminal points of integration only. Line of integration is semicircle of radius 1 centered at point (0; 0), located to the right of *y*-axis. Integration is performed in positive direction, then variable *z* changes from initial value $z_1=-i$ to the terminal value $z_2=i$. So, we have

$$\int_L \left(4z^3+\cos 2z\right)dz=\int_{-i}^{i}\left(4z^3+\cos 2z\right)dz=\left(z^4+\frac{1}{2}\sin 2z\right)\Bigg|_{-i}^{i}=$$

$$=\left(i^4+\frac{1}{2}\sin 2i\right)-\left((-i)^4+\frac{1}{2}\sin(-2i)\right)=1+\frac{i}{2}\operatorname{sh}2-1+\frac{i}{2}\operatorname{sh}2=i\operatorname{sh}2.$$

Example 4. Evaluate the integrals:

a) $\displaystyle\oint_{|z+3i|=2}\frac{e^z dz}{z(z-2i)}$; b) $\displaystyle\oint_{|z-3i|=2}\frac{e^z dz}{z(z-2i)}$; c) $\displaystyle\oint_{|z|=1}\frac{\cos z dz}{z^2(z-2i)}$.

Solution. We have integrals along closed contours. Then we apply Cauchy's integral formula (or theorem).

a) line of integration given by equation $|z+3i|=2$ is a circle with radius 2 centered at point (0; –3). Integrand $f(z)=\dfrac{e^z}{z(z-2i)}$ is analytic inside this circle. Then according to Cauchy's integral theorem it follows $\oint\limits_{|z+3i|=2}\dfrac{e^z dz}{z(z-2i)}=0$;

b) among singular points $z_1=0$ and $z_2=2i$ of a function $f(z)=\dfrac{e^z}{z(z-2i)}$ the only point $z_2=2i$ is located inside the circle $|z-3i|=2$ (centered at point (0; 3), radius 2) . By the formula (4.2) we have

$$\oint\limits_{|z-3i|=2}\frac{e^z dz}{z(z-2i)}=\oint\limits_{|z-3i|=2}\frac{\frac{e^z}{z}dz}{z-2i}=2\pi i\cdot\underbrace{\left.\frac{e^z}{z}\right|_{z=2i}}_{f(z_0)}=$$

$$=2\pi i\cdot\frac{e^{2i}}{2i}=\pi e^{2i}=\pi(\cos 2+i\sin 2);$$

c) among singular points $z_1=0$ i $z_2=2i$ of a function $f(z)=\dfrac{\cos z}{z^2(z-2i)}$ the only point $z_1=0$ is located inside the circle $|z|=1$ whose multiplicity 2.By the formula (4.3) we have

$$\oint\limits_{|z|=1}\frac{\cos z dz}{z^2(z-2i)}=\oint\limits_{|z|=1}\frac{\overbrace{\left(\frac{\cos z}{z-2i}\right)}^{f(z)}}{(z-0)^{1+1}}dz=\frac{2\pi i}{1!}\cdot\underbrace{\left.\left(\frac{\cos z}{z-2i}\right)'\right|_{z=0}}_{f'(0)}=$$

$$=2\pi i\left.\left(\frac{-\sin z(z-2i)-\cos z}{(z-2i)^2}\right)\right|_{z=0}=\frac{\pi i}{2}.$$

Self-test questions

1. Give the definition of integral of function of a complex variable.
2. How the analytic function is integrated?
3. Formulate the Cauchy's integral theorem.
4. Write Cauchy's integral formula.
5. How Cauchy's integral formula can be used for evaluation of entegrals along closed contour.

Self-test assignments

Task 1. Evaluate the integral $\int\limits_L (1+2iz-4\overline{z})dz$ along lines connecting points $z_1=0$ and $z_2=1+i$ in direction from z_1 to z_2:

a) segment;

b) broken line $z_1z_3z_2$ where $z_3=i$.

Task 2. Evaluate the integral

$$\int\limits_L \left(|z|+2\overline{z}\right)dz; \quad L:\{|z|=3,\ \frac{\pi}{2}\le \arg z\le \pi\}.$$

Task 3. Evaluate the integrals:

a) $\int\limits_{1+i}^{-1-i}(2z+1)dz$ b) $\int\limits_{1+i}^{2i} ze^{z^2/2}dz;$; c) $\int\limits_0^i z\sin z\,dz$.

Task 4. Evaluate the integrals:

a) $\oint\limits_{|z|=0,5}\frac{dz}{z(z^2+1)}$; b) $\oint\limits_{|z|=2}\frac{z+1}{(z-1)^2}dz$; c) $\oint\limits_{|z|=3}\frac{\operatorname{sh}(i\pi z)}{(z+2)^4}dz$.

Answers: 1. a) $-5+i$; b) $-5+5i$. **2.** $-9+9i(\pi-1)$. **3.** a) $-2(1+i)$; b) $e^{-2}-\cos 1-i\sin 1$; c) $1+i(\operatorname{sh}1-\operatorname{ch}1)$. **4.** a) $2\pi i$; b) $2\pi i$; c) $\frac{\pi^4}{3}$.

Topic 5. TAYLOR'S AND LAURENT SERIES. ISOLATED SINGULAR POINTS.

Plan

1. Taylor's and Laurent series.
2. Isolated singular points, their classification.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 5 the student should ***know***: definition of Taylor's and Laurent series, difference between them, types of isolated singular points; ***be able to:*** expand the functions into Taylor's and Laurent series, find isolated singular points and establish their type.

The basic theoretical information

Taylor's and Lourant series. Power series of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

is called *Taylor's series* of a function $f(z)$ at neighborhood of a point z_0.

The following expansions of elementary functions are true:

1) $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots + (-1)^n z^n + \ldots$ $\quad (|z| < 1)$; (5.1)

2) $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots$ $\quad (|z| < 1)$; (5.2)

3) $e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots + \dfrac{z^n}{n!} + \ldots$ $\quad (z \in C)$;

4) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots + (-1)^n \dfrac{z^{2n+1}}{(2n+1)!} + \ldots$ $\quad (z \in C)$; (5.3)

5) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots + (-1)^n \dfrac{z^{2n}}{(2n)!} + \ldots$ $\quad (z \in C)$. (5.4)

Any function $f(z)$, analytic in the annulus $r < |z - z_0| < R$ can be expanded in Laurent series:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z^n) = \underbrace{\sum_{n=0}^{+\infty} a_n (z - z^n)}_{\textit{true part}} + \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{\textit{principal part}}$$

Isolated singular points, their classification. If a function $f(z)$ fails to be analytic at the point z_0 then the point z_0 is called *singular*.

The point z_0 is called *isolated* if there is no any other singular point at the neighborhood of point z_0.

The classification of isolated singular points:

1) z_0 is *removable* singular point ($\lim\limits_{z\to z_0} f(z) = a_0$, Laurent series doesn't contain negative powers of $z - z_0$);

2) z_0 is a *pole* ($\lim\limits_{z\to z_0} f(z) = \infty$, Laurent series contains finite number of negative powers $z - z_0$);

3) z_0 is *essential* point ($\lim\limits_{z\to z_0} f(z)$ doesn't exist, Laurent series contains infinite number of negative powers $z - z_0$).

The point z_0 is called a *zero* or a *root* of a function $f(z)$ of order *m* if the following conditions are hold

$$f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0,\ f^{(m)}(z_0) \neq 0.$$

If *m* = 1 then the point z_0 is called *simple* zero.

If z_0 is pole of *m-th* order of a function $f(z)$ then z_0 is zero of *m-th* order of a function $\dfrac{1}{f(z)}$ and vice versa if z_0 is zero of *m-th* order of a function $f(z)$ then z_0 is the pole of *m-th* order of a function $\dfrac{1}{f(z)}$.

Examples of solution of typical problems

Example 1. Expand the function $f(z) = \dfrac{1}{3-2z}$ in powers $z - z_0$: a) in Taylor's series if $z_0 = 0$; b) in Taylor's series if $z_0 = 4$; c) in Laurent series if $z_0 = 0$.

Solution. a) the function $f(z)$ has one singular point $z = \dfrac{3}{2}$. Hence, the function is analytic in the circle $|z| < \dfrac{3}{2}$ (Fig.5.1, *a*) and it can be expanded inside the circle in Taylor's series $\dfrac{1}{3-2z} = \sum\limits_{n=0}^{\infty} a_n z^n$.

So, we obtain

$$f(z)=\frac{1}{3}\cdot\frac{1}{1-\frac{2z}{3}}=\frac{1}{3}\underbrace{\left(1+\frac{2z}{3}+\left(\frac{2z}{3}\right)^2+\ldots\right)}_{by\,the\;formula\,(5.2)}=\sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}}z^n;$$

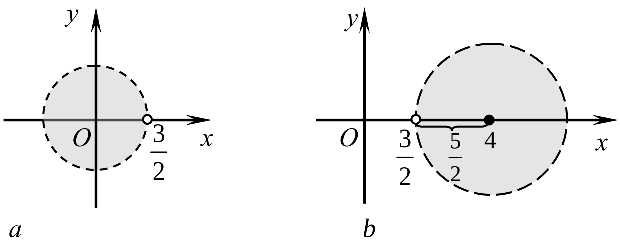


Fig. 5.1

b) let’s perform the transformations

$$f(z)=\frac{1}{3-2z}=\frac{1}{-2(z-4)-5}=-\frac{1}{5}\cdot\frac{1}{1+\frac{2(z-4)}{5}}=$$

$$=-\frac{1}{5}\underbrace{\left(1-\frac{2(z-4)}{5}+\frac{2^2(z-4)^2}{5^2}-\frac{2^3(z-4)^3}{5^3}+\ldots\right)}_{by\,the\;formula\,(5.1)}=\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{2^n(z-4)^n}{5^{n+1}}.$$

The domain of convergence of power series is the circle $|z-4|<\frac{5}{2}$ (Fig.5.1, *b*);

c) the circle $|z|=\frac{3}{2}$ passing through singular point $z=\frac{3}{2}$ devides the plane by two parts. Inside the circle $|z|<\frac{3}{2}$ the function $f(z)$ can be expanded in Taylor’s series (see issue a)); in domain $|z|>\frac{3}{2}$ (outside the circle i.e. neighborhood of infinite point $z=\infty$) the function $f(z)$ can be expanded in Laurent series with negative powers *z*. So,

$$f(z)=\frac{1}{3-2z}=\frac{1}{-2z}\cdot\frac{1}{1-\frac{3}{2z}}=\frac{1}{-2z}\underbrace{\left(1+\frac{3}{2z}+\left(\frac{3}{2z}\right)^2+\ldots\right)}_{by\,the\;formula\,(5.2)}=-\sum_{n=0}^{\infty}\frac{3^n}{2^{n+1}z^{n+1}}.$$

Example 2. Find the zeros of the function $f(z)=1-\cos 2z$ and determine their order.

Solution. The zeros of this function are the roots of the equation $1-\cos 2z=0$:

$$z_n=\pi n \quad (\mathrm{n}=0,\pm 1,...).$$

We determine their order as follows:

$$f'(z)=2\sin 2z,\ f'(\pi n)=2\sin 2\pi n=0,$$

$$f''(z)=4\cos 2z,\ f''(\pi n)=4\cos 2\pi n=1\neq 0\ .$$

Hence, all points $z_n=\pi n$, $n\in Z$ are zeros of second order of given function.

Example 3. Find singular points of a function $f(z)$ and determine their character:

a) $f(z)=\dfrac{e^{2z}-e^{z}}{z}$; b) $f(z)=\dfrac{e^{3z}+2}{z^3}$; c) $f(z)=\dfrac{1-\cos 2z}{z^3}$;

d) $f(z)=z^2\sin\dfrac{1}{z}$; e) $f(z)=\operatorname{ctg} z$; f) $f(z)=\dfrac{z^2}{(z^2-z-2)(z+1)}$.

Solution. In the issues a) – d) the point $z_0=0$ is a singular point.

a) calculate the limit

$$\lim_{z\to 0}f(z)=\lim_{z\to 0}\frac{e^{2z}-e^{z}}{z}=\lim_{z\to 0}\frac{e^{z}(e^{z}-1)}{z}=\lim_{z\to 0}e^{z}\cdot\lim_{z\to 0}\frac{e^{z}-1}{z}=e^{0}\cdot 1=1.$$

Hence, $z_0=0$ is a removable singular point;

b) since $\lim\limits_{z\to 0}\dfrac{e^{3z}+2}{z^3}=\infty$ then the point $z_0=0$ is a pole of this function. For the function $\varphi(z)=\dfrac{1}{f(z)}=\dfrac{z^3}{e^{3z}+2}$ the point $z_0=0$ is zero of third order, hence, $z_0=0$ is a pole of the third order of function $f(z)$;

c) the first way. Let’s find the limit

$$\lim_{z\to 0}\frac{1-\cos 2z}{z^3}=\lim_{z\to 0}\frac{2\sin^2 z}{z^3}=\left|\sin z\equiv z\right|=\lim_{z\to 0}\frac{2z^2}{z^3}=\lim_{z\to 0}\frac{2}{z}=\infty .$$

Hence, $z_0=0$ is a pole (as at previous example) but not of the third order but of the first order, i.e. $z_0=0$ is a simple pole.

The second way.

$$f(z)=\frac{1-\cos 2z}{z^3}=\frac{1-\underbrace{\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}\ldots\right)}_{by\,the\;formula\,(5.4)}}{z^3}=$$

$$=\frac{4}{2!\cdot z}-\frac{16z}{4!}+\frac{64z^3}{6!}-\ldots .$$

Existence of term $\frac{2}{z^1}=2z^{-1}$ (moreover it is a unique term with negative power z) points that $z_0=0$ is a simple pole;

d) let’s expend the function $f(z)$ in Laurent series at the neighborhood of a point $z_0=0$:

$$f(z)=z^2\sin\frac{1}{z}=z^2\left(\frac{1}{z}-\frac{1}{3!\cdot z^3}+\frac{1}{5!\cdot z^5}-\ldots\right)=z-\frac{1}{3!\cdot z}+\frac{1}{5!\cdot z^3}-\ldots .$$

This expansion contains infinite number of negative powers z. Therefor the point $z_0=0$ is a essential singular point;

e) $f(z)=\mathrm{ctg}\,z=\frac{\cos z}{\sin z}$. Singular points of this function are the roots of the equation $\sin z=0$, i.e. infinite set of values $z_n=\pi n$, $n\in Z$. All there points are poles of the first order. Really, since $\cos z_n=(-1)^n\neq 0$, then it is enough to define the order of zero z_n of the function $\varphi(z)=\sin z$: $\varphi'(z)=\cos z$, $\varphi'(z_n)=\cos(\pi n)\neq 0$. Hence, all points $z_n=\pi n$, $n\in Z$ are the first order zeros of the function $\varphi(z)$ and respectively the first order poles of a given function $f(z)$;

f) $f(z)=\dfrac{z^2}{(z^2-z-2)(z+1)}=\dfrac{z^2}{(z-2)(z+1)^2}$. Singular points are $z=2$ and $z=-1$ moreover $z=2$ is a simply poles and $z=-1$ is the second order poles.

Self-test questions

1. What is Taylor's series of a function $f(z)$?
2. What is Laurent series of a function $f(z)$?
3. What parts does Laurent series consist of?
4. What formulas are used for expansion of the function in Taylor's or Laurent series ?
5. What is isolated singular point?
6. How the isolated singular points are classified?

Self-test assignments

Task 1. Expand the function $f(z)=\dfrac{1}{2+5z}$ by the power $z-z_0$: a)in Taylor's series if $z_0=0$; b) in Taylor's series if $z_0=-1$; c) in Laurent series if $z_0=0$.

Task 2. Expand the function $f(z)=\dfrac{1}{(z-2)(z-3)}$ in Laurent series by the powers z in annulus: a) $2<|z|<3$; b) $|z|>3$.

Task 3. Find singular points of the function and determine their types:

a) $f(z)=\dfrac{1-e^{2z}}{z}$; b) $f(z)=\dfrac{z^2+1}{z^4}$; c) $f(z)=\dfrac{\sin 2z}{z^4}$;

d) $f(z)=e^{\frac{1}{z}}$; e) $f(z)=\mathrm{tg}^2 z$; f) $f(z)=\dfrac{\sin^2 \pi z}{(z^2-4)(z+2)^2}$.

Answers: 1. a) $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}\cdot 5^n}{2^{n+1}}z^n$, $|z|<\dfrac{2}{5}$; b) $-\sum\limits_{n=0}^{\infty}\dfrac{5^n(z+1)^n}{3^{n+1}}$, $|z+1|<\dfrac{3}{5}$;

c) $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}2^n}{5^{n+1}z^{n+1}}$, $|z|>\dfrac{2}{5}$. **2.** a) $-\sum\limits_{n=1}^{\infty}\dfrac{2^{n-1}}{z^n}-\dfrac{1}{3}\sum\limits_{n=0}^{\infty}\left(\dfrac{z}{3}\right)^n$; b) $\sum\limits_{n=1}^{\infty}\dfrac{3^{n-1}-2^{n-1}}{z^n}$.

3. a) $z_0 = 0$ is a removable point; b) $z_0 = 0$ is a pole of 4th order; c) $z_0 = 0$ is a pole of 3d order; d) $z_0 = 0$ is an essential singular point; e) $z_n = \frac{\pi}{2} + \pi n$, where n = 0, ±1, ±2,… are poles of 2d order; f) $z = 2$ is a removable singular point, $z = -2$ is a simple pole.

Topic 6. RESIDUES. APPLICATION OF RESIDUES TO CALCULATION OF INTEGRALS

Plan

1. Residue of function.
2. Application of residues to calculation of integrals.

Literature: [1] — [5].

Methodical guidelines

After studying the material of topic 6 the student should ***know***: definition of residue, formulas for searching of residue for different types of singular points; **be able to:** find residues by the correspondent limits and Laurent series, use residues to calculation of integrals.

The basic theoretical information

Residue of the function. Let z_0 be isolated singular point of single-valued function $f(z)$, L is a closed contour oriented counterclockwise and such that the point z_0 is contained inside L. Moreover L doesn't include other singular points.

Residue of the function $f(z)$ at a point $z = z_0$ ($\operatorname*{Res}_{z=z_0} f(z)$) is the integral

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_L f(z)dz.$$

If z_0 is a regular or finite removable singular point of the function $f(z)$ then $\operatorname*{Res}_{z=z_0} f(z) = 0$.

The residue of the function $f(z)$ at the point z_0 is equal to the coefficient a_{-1} at $\frac{1}{z-z_0}$ in expansion of the function $f(z)$ in Laurent series at the neighborhood of point z_0:

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1}$$

If z_0 is a simple pole of the function $f(z)$ then

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} f(z)(z-z_0).$$

If $f(z) = \frac{\varphi(z)}{\psi(z)}$ where $\varphi(z_0) \neq 0$, $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, i.e. z_0 is a simple pole of the function $f(z)$ then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{6.1}$$

If z_0 is a pole of m^{th} order then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}((z-z_0)^m f(z))}{dz^{m-1}}. \tag{6.2}$$

Application of residues to calculation of integrals.

1. *The basic theorem about residue*s. Let the function $f(z)$ be analytic in closed domain *D* with boundary *L*, except finite number of singular points $z_1, z_2, \ldots, z_N$, located inside domain *D*.

Then

$$\oint_{L^+} f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} f(z). \tag{6.3 }$$

2. Integral of the form

$$\int_0^{2\pi} R(\cos t, \sin t)dt,$$

where $R(u, \upsilon)$ is rational function of two variables u and υ, moreover $R(\cos t, \sin t)$ is continuous at the interval $[0; 2\pi]$, can be evaluated by means of theorem about residues.

Let's introduce new complex variable $z = e^{it}$, then

$$dt = \frac{dz}{iz}, \cos t = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

As a result of such substitution the interval $t\in[0;2\pi]$ is reflected in the circle $|z|=1$. Hence,

$$\int_0^{2\pi} R(\cos t,\,\sin t)dt=\oint_{|z|=1}\tilde{R}(z)dz=2\pi i\sum_{k=1}^{N}\operatorname*{Res}_{z=z_k}\tilde{R}(z) \qquad (6.4)$$

where $\tilde{R}(z)$ is rational fractional function of variable z, the traversing along the circle $|z|=1$ is counterclockwise.

Examples of solution of typical problems

Example 1. Find residues of the functions at finite singular points:

a) $f(z)=\dfrac{e^{2z}-e^{z}}{z}$; b) $f(z)=\dfrac{e^{3z}+2}{z^3}$; c) $f(z)=\dfrac{1-\cos 2z}{z^3}$;

d) $f(z)=z\sin\dfrac{1}{z}$; e) $f(z)=\operatorname{ctg} z$; f) $f(z)=\dfrac{z^2}{(z-2)(z+1)^2}$.

Solution. In example 3 (see page 30) singular points of these functions are defined and their types are established.

We have:

a) $z_0=0$ is a removable singular point, then $\operatorname*{Res}_{z=0}\dfrac{e^{2z}-e^{z}}{z}=0$;

b) $z_0=0$ is the 3[d] order pole of the function $f(z)=\dfrac{e^{2z}+2}{z^3}$ then by the formula (6.2) where $m=3$, we have

$$\operatorname*{Res}_{z=0} f(z)=\frac{1}{2!}\lim_{z\to 0}\frac{d^2(z^3 f(z))}{dz^2}=\frac{1}{2}\lim_{z\to 0}\left(z^3\cdot\frac{e^{3z}+2}{z^3}\right)''=$$

$$=\frac{1}{2}\lim_{z\to 0}\left(e^{3z}+2\right)''=\frac{1}{2}\lim_{z\to 0}\left(9e^{3z}\right)=4{,}5;$$

c) $z_0=0$ is a simple pole then $\operatorname*{Res}_{z=0} f(z)=\lim\limits_{z\to 0}(zf(z))=$

$$=\lim_{z\to 0}\left(z\cdot\frac{1-\cos 2z}{z^3}\right)=\lim_{z\to 0}\frac{1-\cos 2z}{z^2}=\lim_{z\to 0}\frac{2\sin^2 z}{z^2}=2;$$

d) $z_0=0$ is essential singular point. Since

$$z^2\sin\frac{1}{z}=z-\frac{1}{6z}+\frac{1}{120z^3}-\ldots=z+\underbrace{\left(-\frac{1}{6}\right)}_{a_{-1}}\frac{1}{z}+\frac{1}{120z^3}-\ldots,$$

then $\operatorname*{Res}_{z=0} f(z)=a_{-1}=-\dfrac{1}{6}$;

e) $z_n=\pi n$, $n\in Z$ are simple poles. Then

$$\operatorname*{Res}_{z=\pi n} f(z)=\lim_{z\to\pi n}((z-\pi n)\cdot\operatorname{ctg} z)=\left|\begin{array}{l} z-\pi n=t \\ t\to 0\end{array}\right|=$$

$$=\lim_{t\to0}(t\cdot\operatorname{ctg}(\pi n+t))=\lim_{t\to0}(t\cdot\operatorname{ctg} t)=\lim_{t\to0}\left(\frac{t}{\sin t}\cdot\cos t\right)=1;$$

f) $z_1=2$ is a simple pole, and $z_2=-1$ is the 2[d] order pole of function $f(z)$. Then according to formula (6.2) we obtain

$$\operatorname*{Res}_{z=2} f(z)=\lim_{z\to2}\left((z-2)\cdot\frac{z^2}{(z-2)(z+1)^2}\right)=\lim_{z\to2}\frac{z^2}{(z+1)^2}=\frac{4}{9}.$$

We obtain the same result if formula (6.1) is used :

$$\varphi(z)=z^2,\ \psi(z)=(z-2)(z+1)^2,\varphi(2)=4\neq0,\psi(2)=0,$$
$$\psi'(z)=(z+1)^2+2(z-2)(z+1),\psi'(2)=9+0=9\neq0.$$

Hence, $$\operatorname*{Res}_{z=2} f(z)=\frac{\varphi(2)}{\psi'(2)}=\frac{4}{9};$$

$$\operatorname*{Res}_{z=-1} f(z)\overset{(6.2),m=2}{=}\frac{1}{(2-1)!}\lim_{z\to-1}\left((z+1)^2\cdot\frac{z^2}{(z-2)(z+1)^2}\right)'=$$

$$=\lim_{z\to-1}\left(\frac{z^2}{z-2}\right)'=\lim_{z\to-1}\left(\frac{2z(z-2)-z^2\cdot1}{(z-2)^2}\right)=\frac{6-1}{9}=\frac{5}{9}.$$

Example 2. Evaluate the integrals:

a) $\displaystyle\oint_{|z|=3}\frac{e^{2z}-e^z}{z}dz$; b) $\displaystyle\oint_{|z|=3}\frac{e^{3z}+2}{z^3}dz$; c) $\displaystyle\oint_{|z|=3}\frac{1-\cos2z}{z^3}dz$;

d) $\displaystyle\oint_{|z|=3}z\sin\frac{1}{z}dz$; e) $\displaystyle\oint_{|z|=3}\operatorname{ctg} z\,dz$; f) $\displaystyle\oint_{|z|=3}\frac{z^2}{(z-2)(z+1)^2}dz$.

Solution. In all cases, the integration contour is a circle of radius 3 with the center at the point (0; 0), which is traversed counter-clockwise. So, we have: a) $\oint\limits_{|z|=3} \frac{e^{2z}-e^{z}}{z}dz = 2\pi i \operatorname*{Res}\limits_{z=0} \frac{e^{2z}-e^{z}}{z} = 2\pi i \cdot 0 = 0$;

b) $\oint\limits_{|z|=3} \frac{e^{3z}+2}{z^3}dz = 2\pi i \operatorname*{Res}\limits_{z=0} \frac{e^{3z}+2}{z^3} = 2\pi i \cdot 4{,}5 = 9\pi i$;

c) $\oint\limits_{|z|=3} \frac{1-\cos 2z}{z^3}dz = 2\pi i \operatorname*{Res}\limits_{z=0} \frac{1-\cos 2z}{z^3} = 2\pi i \cdot 2 = 4\pi i$;

d) $\oint\limits_{|z|=3} z\sin\frac{1}{z}dz = 2\pi i \operatorname*{Res}\limits_{z=0} z\sin\frac{1}{z} = 2\pi i \cdot \left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$;

e) only one singular point $z=0$ occures inside the circle $|z|=3$ then $\oint\limits_{|z|=3} \operatorname{ctg} z dz = 2\pi i \operatorname*{Res}\limits_{z=0} \operatorname{ctg} z = 2\pi i \cdot 1 = 2\pi i$;

f) two singular points $z_1 = 2$ and $z_2 = -1$ occure inside the circle $|z|=3$. Then according to the formula (6.3) we obtain

$$\oint\limits_{|z|=3} \frac{z^2}{(z-2)(z+1)^2}dz = 2\pi i\left(\operatorname*{Res}\limits_{z=2} \frac{z^2}{(z-2)(z+1)^2} + \right.$$

$$\left. + \operatorname*{Res}\limits_{z=-1} \frac{z^2}{(z-2)(z+1)^2}\right) = 2\pi i \cdot \left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i .$$

Self-test questions

1. What is the residue of the function of a complex variable at an isolated singular point?
2. How to determine the residue of a function by its Laurent series?
3. What does equal the residue of the function at the removable singular point?
4. What formulas are used to determine the residue of a function at a point if it is: a) a simple pole; b) the pole of multiplicity m?
5. What formulas are used to determine the residue of a function at a point if it is: a) a simple pole; b) the pole of multiplicity m?
6. Formulate the basic theorem about remainders.

Self-test assignments

Task 1. Find the residues of functions at finite singular points:

a) $f(z)=\dfrac{1-e^{2z}}{z}$; b) $f(z)=\dfrac{z^2+1}{z^4}$; c) $f(z)=\dfrac{\sin 2z}{z^4}$; d) $f(z)=e^{\frac{1}{z}}$;

e) $f(z)=\operatorname{tg}^2 z$; f) $f(z)=\dfrac{\sin^2 \pi z}{(z^2-4)(z+2)^2}$; g) $f(z)=\dfrac{2}{z}+\dfrac{3}{z^3}-z$.

Task 2. Evaluate the integrals: a) $\oint\limits_{|z|=2}\dfrac{1-e^{2z}}{z}dz$; b) $\oint\limits_{|z|=2}\dfrac{z^2+1}{z^4}dz$;

c) $\oint\limits_{|z|=2}\dfrac{\sin 2z}{z^4}dz$; d) $\oint\limits_{|z|=2} e^{\frac{1}{z}}dz$; e) $\oint\limits_{|z|=2}\operatorname{tg}^2 z\,dz$; f) $\oint\limits_{|z+1|=2}\dfrac{\sin^2 \pi z}{(z^2-4)(z+2)^2}dz$;

g) $\oint\limits_{|z|=2}\left(\dfrac{2}{z}+\dfrac{3}{z^3}-z\right)dz$; h) $\oint\limits_{|z|=3}\dfrac{\sin z}{(z-2i)^3}dz$.

Task 3. Evaluate the integrals: a) $\int\limits_0^{2\pi}\dfrac{dt}{4\sin t+5}$; b) $\int\limits_0^{2\pi}\dfrac{dt}{2\cos t+7}$.

Answers: 1. a) 0; b) 0; c) $-\dfrac{4}{3}$; d) 1; e) 1 at each point $z_n=\dfrac{\pi}{2}+\pi n, n\in Z$;

f) $\operatorname*{Res}\limits_{z=2} f(z)=0$, $\operatorname*{Res}\limits_{z=-1} f(z)=-\dfrac{\pi^2}{4}$; g) $\operatorname*{Res}\limits_{z=0} f(z)=2$. **2.** a) 0; b) 0; c) $-\dfrac{8\pi i}{3}$;

d) $2\pi i$; e) $4\pi i$; f) $-\dfrac{\pi^3 i}{2}$; g) $\dfrac{\pi i}{2}$; h) $-\dfrac{i}{2}\operatorname{sh}2$. 3. a) $\dfrac{2\pi}{3}$; b) $\dfrac{2\sqrt{5}\pi}{15}$.

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Contributors:
LASTIVKA Ivan Oleksiyovich
LIASHENKO Yana Grygorivna
REPETA Viktor Kuzmych